GENERIC ORBITS AND TYPE ISOLATION IN THE GURARIJ SPACE

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Abstract. We study model-theoretic aspects of the separable Gurarij space $\mathbf{G}$, in particular type isolation and the existence of prime models, without use of formal logic.

(i) If E is a finite-dimensional Banach space, then the set of isolated types over E is dense, and there exists a prime Gurarij over E. This is the unique separable Gurarij space $\mathbf{G}$ extending E with the unique Hahn-Banach extension property (*property* U), and the orbit of $\mathrm{id}\colon E \hookrightarrow \mathbf{G}$ under the action of $\mathrm{Aut}(\mathbf{G})$ is a dense G_δ in the space of all linear isometric embeddings $E \hookrightarrow \mathbf{G}$.

(ii) If E is infinite-dimensional then there are no non realised isolated types, and therefore no prime model over E (unless $\mathbf{G} \cong E$), and all orbits of embeddings $E \hookrightarrow \mathbf{G}$ are meagre. On the other hand, there are Gurarij spaces extending E with property U.

We also point out that the class of Gurarij space is the class of models of an $\aleph_0$-categorical theory with quantifier elimination, and calculate the density character of the space of types over E, answering a question of Avilés et al.

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Introduction

In 1966, Gurarij [Gur66] defined what came to be known as the *(separable) Gurarij space*, and proved that it almost isometrically unique. The isometric uniqueness of the Gurarij space was only proved in 1976 by Lusky [Lus76]. In the same paper, Lusky points out that the arguments could be modified to prove also the isometric uniqueness of the separable Gurarij space equipped with a distinguished smooth unit vector. In other words, if $\mathbf{G}$ denotes the separable Gurarij space, then the set of smooth unit vectors in $\mathbf{G}$ forms an orbit under the action of the linear isometry group $\mathrm{Aut}(\mathbf{G})$. By Mazur [Maz33], this orbit is moreover a dense G_δ subset of the unit sphere.

These facts are strongly reminiscent of model theoretic phenomena, and indeed turn out to be special cases of such. It was observed some time ago by the second author that the uniqueness of the Gurarij space can be accounted for as it being the unique separable model of an $\aleph_0$-categorical theory, which moreover eliminates quantifiers. Similarly, the Gurarij space is atomic over a vector if and only if the latter is smooth, so Lusky's second uniqueness result is a special case of the uniqueness of the prime model (namely, separable atomic model) over a vector of norm one (by quantifier elimination, the type of a single vector is entirely determined by its norm).

These observations serve as a starting point for the present paper, whose goals are threefold:

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- Our primary goal is to make the observations above precise, and generalise them to uniqueness results over a subset other than the empty set or a singleton – in other words, we study uniqueness and primeness of the Gurarij space over a subspace E of dimension possibly greater than one. We prove that if $\dim E < \infty$ then essentially the same holds as when $\dim E = 1$, that is to say that, first, there exists a linear isometric embedding of E in $\mathbf{G}$ over whose image $\mathbf{G}$ is atomic, and second, that the set of such embeddings forms a dense G_δ orbit among all linear isometric embeddings of E in $\mathbf{G}$. Moreover, $\mathbf{G}$ is atomic over (a copy of) E if and only if E is "smooth" in $\mathbf{G}$, that is to say that the Hahn-Banach extension of linear functionals from E to $\mathbf{G}$ is unique (we shall say that E has *property* U in $\mathbf{G}$). When $\dim E = \infty$, we show that the only infinite-dimensional subspace of $\mathbf{G}$ over which $\mathbf{G}$ is atomic is $\mathbf{G}$ itself, and all orbits are meagre. On the other hand, any separable E admits an isometric linear embedding in $\mathbf{G}$ with property U (just that if $\dim E = \infty$, these embeddings need not form an orbit).
- A secondary goal is to present a subset of the toolbox of model theory in a manner accessible non logicians. Starting with a definition of types and type spaces which does not make any use of formal logic, we discuss general topics such as type isolation, the Tarski-Vaught Criterion, the Omitting Types Theorem, and the primeness and uniqueness of atomic models. While we do this in a fairly specific context, we attempt to present arguments which would be valid in the general case (possibly with separate follow-up results which improve the general ones in a manner specific to the context of the Gurarij space). There are few results which make explicit use of formal logic (essentially, Proposition 1.19 and Theorem 2.3), which serve mostly as parenthetical remarks required for completeness, and are not used in any way in other parts of the paper.
- A minor tertiary goal is to present to model theorists, who are familiar with the tools mentioned in the previous item *in the context of classical logic*, how these tools adapt to the metric setting.

In Section 1 we define (quantifier-free) types and type spaces over a Banach space E, and study their properties. The topometric structure of the type space, a fundamental notion of metric model theory, is defined there, as well as (topometrically) isolated types, which are one of the main objects of study of this paper.

In Section 2 we start studying Gurarij spaces. At the technical level, we define and study Gurarij (and other) spaces which are atomic over a fixed separable parameter space E, and prove the Omitting Types Theorem (Theorem 2.11). We prove appropriate generalisations of the homogeneity and universality properties of the Gurarij space to homogeneity and universality over E. In particular, we show that the prime Gurarij spaces over E (see Corollary 2.12) are those Gurarij space which are separable and atomic over E, and that they are all isometrically isomorphic over E, denoted $\mathbf{G}[E]$. We also give the standard model theoretic criterion for the existence of $\mathbf{G}[E]$.

At this point we move on to the question of when the isolated types over E are dense, and how to characterise them in a more Banach theoretic fashion. In Section 3 we consider the particularly easy case where $\dim E = 1$, which serves as a good indication for where to look later on. Before considering the general case, we introduce an essential tool in Section 4, namely the presentation of 1-types as convex Katětov functions (as per [Benb]), and the Legendre-Fenchel transformation of these. This tool allows us to prove in Section 5 that if $\dim E < \infty$ then $\mathbf{G}[E]$ exists, and that a type is isolated if and only if it (generates an extension which) has property U. The same tool is used in Section 6 to show that if $\dim E = \infty$ then the situation is quite different: there are no isolated types other than the realised ones, and yet the types with property U are dense. The description of generic orbits alluded to above follows.

We conclude in Section 7 with a "counting types" result, showing that the space of types over E is metrically separable if and only if E is finite-dimensional and polyhedral. This allows us to answer a question of Avilés et al. [ACC+11].

All Banach spaces under consideration are over the real numbers, denoted E, F, and so on. An embedding (or isomorphism, automorphism) of Banach spaces is always isometric. Linear maps will simply be referred to as such, even when they are homeomorphic.

The topological dual of a Banach space E will be denoted E^*. We shall often use the notation $E_{\leq 1}$ for the closed unit ball of E, $E_{=1}$ for the unit sphere, and so on.

1. Quantifier-free types in Banach spaces

Before we start, let us state the following basic amalgamation result which we shall use many times, quite often implicitly.

Fact 1.1. *For any three Banach spaces E, F_0 and F_1, and isometric embeddings $f_i\colon E \to F_i$, there is a third Banach space G and isometric embeddings $g_i\colon F_i \to G$ such that $g_0 f_0 = g_1 f_1$.*

Proof. Equip the direct sum $F_0 \oplus F_1$ with the semi-norm $\|v+u\| = \inf_{w\in E} \|v+w\| + \|u-w\|$, divide by the kernel and complete. $\blacksquare_{1.1}$

We can now define the fundamental objects of study of this section, and, to a large extent, the entire paper.

Definition 1.2. Let E be a Banach space and X a sequence of symbols which we call *variables*. We let $E(X) = E \oplus \bigoplus_{x\in X} \mathbf{R}x$, and define $\mathrm{S}_X(E)$ to consist of all semi-norms on $E(X)$ which extend the norm on E, calling it the *space of types in X over E*. We shall denote members of $\mathrm{S}_X(E)$ by ξ, ζ and so on, and the corresponding semi-norms by $\|\cdot\|^\xi$, $\|\cdot\|^\zeta$ and so on (model-theoretic tradition would have us denote types by p, q and so on, but an expression such as $\|\cdot\|^p$ may be disastrously suggestive of a meaning other than the intended one).

Quite often X will be of the form $\{x_i\}_{i\in I}$ for some index set I, in which case we write $E(I) = E \oplus \bigoplus_{i\in I} \mathbf{R}x_i$ instead of $E(X)$, and similarly $\mathrm{S}_I(E)$, whose members are called I-types.

Definition 1.3. Given a Banach space extension $E \subseteq F$ and an I-sequence $\bar a = \{a_i\}_{i\in I} \subseteq F$, we define its *type over E*, in symbols $\xi = \mathrm{tp}(\bar a/E) \in \mathrm{S}_I(E)$, to be the semi-norm $\|b + \sum \lambda_i x_i\|^\xi = \|b + \sum \lambda_i a_i\|$, and say that $\bar a$ *realises* ξ. When a sequence $\bar b$ generates E, we may also write $\mathrm{tp}(\bar a/\bar b)$ for $\mathrm{tp}(\bar a/E)$.

Conversely, given a type $\xi \in \mathrm{S}_I(E)$, we define the Banach space *generated* by ξ, in symbols $E[\xi]$, as the space obtained from $\bigl(E(I), \|\cdot\|^\xi\bigr)$ by dividing by the kernel and completing, together with the distinguished generators $\{x_i\}_{i\in I} \subseteq E[\xi]$.

Definition 1.4. We equip $\mathrm{S}_I(E)$ with a topological structure as well as with a metric structure *which are often distinct.* The *topology* on $\mathrm{S}_I(E)$ is the least one in which, for every member $x \in E(I)$, the map $\hat x\colon \xi \mapsto \|x\|^\xi$ is continuous. Given $\xi, \zeta \in \mathrm{S}_I(E)$, we define the *distance* $d(\xi,\zeta)$ to be the infimum, over all F extending E and over all realisations $\bar a$ and $\bar b$ of ξ and ζ, respectively, of $\sup_i \|a_i - b_i\|$.

Remark 1.5. A model-theorist will recognise types as we define them here as *quantifier-free* types, which do not, in general, capture "all the pertinent information". However, by Fact 1.1, they do capture a maximal existential type. Moreover, it follows from Lemma 1.14 below (and more specifically, from the assertion that $\pi_{\bar x}\colon \mathrm{S}_{\bar x,y}(0) \to \mathrm{S}_{\bar x}(0)$ is open) that being an existentially closed Banach space is an elementary property, so the theory of Banach spaces admits a model companion. Then Fact 1.1 can be understood to say that the model companion eliminates quantifiers, so quantifier-free types and types are in practice the same. As we shall see later, the model companion is separably categorical, and its unique separable model is $\mathbf{G}$, the separable Gurarij space.

It is fairly clear that the distance refines the topology, and we shall see that unless the parameter space E is trivial and I is finite, they are in fact distinct. In a sense, the distance as defined on $\mathrm{S}_I(E)$ is "incorrect" when I is infinite (for more reasons than the mere fact that this distance can be infinite), and we should never have defined it for such I if not for Proposition 1.7 below holding for infinite I as well.

Lemma 1.6. *Let E, F be Banach spaces, I an index set, and consider tuples $\bar a = (a_i)_{i\in I} \in E^I$, $\bar b \in F^I$ and $\bar\varepsilon \in \mathbf{R}^I$. Let also $\mathbf{R}^{(I)}$ denote the set of all I-tuples in which all but finitely many positions are zero. The following conditions are equivalent.*

(i) *There exists a semi-norm $\|\cdot\|$ on $E \oplus F$ extending the respective norms of E and F, such that for each $i \in I$ one has $\|a_i - b_i\| \le \varepsilon_i$.*
(ii) *For all $\bar r \in \mathbf{R}^{(I)}$, one has*

$$\left| \left\|\sum r_i a_i\right\| - \left\|\sum r_i b_i\right\| \right| \le \sum |r_i|\varepsilon_i.$$

Proof. One direction being trivial, we prove the other. For $c + d \in E \oplus F$ define

$$\|c+d\|' = \inf_{\bar r \in \mathbf{R}^{(I)}} \left\|c - \sum r_i a_i\right\| + \left\|d + \sum r_i b_i\right\| + \sum |r_i|\varepsilon_i.$$

This is easily checked to be a semi-norm, with $\|c\|' \le \|c\|$ for $c \in E$. Now, for $c \in E$ and $\bar r \in \mathbf{R}^{(I)}$ we have

$$\left\|c - \sum r_i a_i\right\| + \left\|\sum r_i b_i\right\| + \sum |r_i|\varepsilon_i \ge \left\|c - \sum r_i a_i\right\| + \left\|\sum r_i a_i\right\| \ge \|c\|.$$

Therefore $\|c\|' = \|c\|$, and similarly $\|d\|' = \|d\|$ for $d \in F$, concluding the proof. $\blacksquare_{1.6}$

Proposition 1.7. *Let $\xi, \zeta \in \mathrm{S}_I(E)$, and let $E(I)_1$ consist of all $a + \sum \lambda_i x_i \in E(I)$ (so $a \in E$ and all but finitely many of the λ_i vanish) such that $\sum |\lambda_i| = 1$. Then*

$$d(\xi, \zeta) = \sup_{x \in E(I)_1} \left| \|x\|^\xi - \|x\|^\zeta \right|.$$

Moreover, the infimum in the definition of distance between types is attained.

Proof. Immediate from Lemma 1.6. $\blacksquare_{1.7}$

Convention 1.8. When referring to the topological or metric structure of $\mathrm{S}_I(E)$, we shall follow the convention that unqualified terms taken from the vocabulary of general topology (open, compact and so on) apply to the topological structure, while terms specific to metric spaces (bounded, complete and so on) refer to the metric structure.

Excluded from this convention is the notion of isolation which will be defined in a manner which takes into account both the topology and the metric.

While this convention may seem confusing at first, it is quite convenient, as in the following.

Lemma 1.9.
(i) *The space $\mathrm{S}_I(E)$ is Hausdorff, and every closed and bounded set thereof is compact.*
(ii) *The distance on $\mathrm{S}_I(E)$ is lower semi-continuous. In particular, the closure of a bounded set is bounded.*
(iii) *Assume that I is finite, say $I = n = \{0, 1, \ldots, n-1\} \in \mathbf{N}$. Then every bounded set is contained in an open bounded set. It follows that the space $\mathrm{S}_n(E)$ is locally compact, and that a compact subset of $\mathrm{S}_n(E)$ is necessarily (closed and) bounded.*
(iv) *A subset $X \subseteq \mathrm{S}_n(E)$ is closed if and only if its intersection with every compact set is compact.*
(v) *Let $m \leq n$, and let $\pi\colon \mathrm{S}_n(E) \to \mathrm{S}_m(E)$ denote the obvious variable restriction map. Then for every $\xi \in \mathrm{S}_n(E)$ and $\zeta \in \mathrm{S}_m(E)$ we have $d(\pi\xi, \zeta) = d(\xi, \pi^{-1}\zeta)$. Moreover there exists $\rho \in \pi^{-1}\zeta$ such that $d(\pi\xi, \zeta) = d(\xi, \rho)$ and $\|x_i\|^\rho = \|x_i\|^\xi$ for all $m \leq i < n$.*
In particular, the map π is metrically open.

Proof. For the first item, clearly $\mathrm{S}_I(E)$ is Hausdorff. If $X \subseteq \mathrm{S}_I(E)$ is bounded, then for every $x \in E(I)$ there exists M_x such that $\|x\|^\xi \leq M_x$ for all $\xi \in X$. We can therefore identify X with a subset of $Y = \prod_x [0, M_x]$, and if X is closed in $\mathrm{S}_I(E)$ then it is closed in Y and therefore compact.

The second item follows from Proposition 1.7, and the third is immediate.

For the fourth item, assume that $X \subseteq \mathrm{S}_n(E)$ is not closed, let $\xi \in \overline{X} \smallsetminus X$ and let U be a bounded neighbourhood of ξ, in which case $\overline{U} \cap X$ is not compact.

For the fifth item, the inequality $\leq$ is immediate. For the opposite inequality, there exists an extension $F \supseteq E$ and realisations $\bar{a}$ of ξ and $\bar{b}$ of ζ in F such that $\|a_i - b_i\| < r$ for $i < m$. Letting $c_i = b_i$ for $i < m$, $c_i = a_i$ for $m \leq i < n$, we see that $\rho = \mathrm{tp}(\bar{c}/E)$ is as desired for both the main assertion and the moreover part. It follows that $\pi B(\xi, r) \supseteq B(\pi\xi, r)$, so π is metrically open. $\blacksquare_{1.9}$

This double structure makes $\mathrm{S}_I(E)$ a *topometric space*, in the sense of [Ben08b].

Definition 1.10. We say that a type $\xi \in \mathrm{S}_n(E)$ is *isolated* if the distance and the topology agree at ξ, i.e., if every metric neighbourhood of ξ is also a topological one.

This is the definition of isolation in a topometric space, taking into account both the metric and the topological structure. Ordinary topological spaces can be viewed as topometric spaces by equipping them with the discrete 0/1 distance, in which case the notion of isolation as defined here coincides with the usual one.

Many results regarding ordinary topological spaces still hold, when translated correctly, with the topometric definitions. For example, the fact that a dense set must contain all isolated points becomes the following. Notice that in Lemma 1.16 below we prove that the set of isolated types it itself metrically closed.

Lemma 1.11. *Let E be a Banach space, $D \subseteq \mathrm{S}_n(E)$ a dense, metrically closed set. Then D contains all isolated types.*

Proof. If ξ is isolated then all metric neighbourhoods of ξ, which are also topological neighbourhoods, must intersect D. $\blacksquare_{1.11}$

One of our aims in this paper is to characterise isolated types. We start with the easiest situation.

Proposition 1.12. *Let* 0 *denote the trivial Banach space. Then every type in* $\mathrm{S}_n(0)$ *is isolated. In other words, the distance on* $\mathrm{S}_n(0)$ *is compatible with the topology.*

Proof. Given $N \in \mathbf{N}$, let $X_N \subseteq 0(n)_1$ be the finite set consisting of all $\sum \lambda_i x_i$ where $\sum |\lambda_i| = 1$ and each λ_i is of the form $\frac{k}{N}$. For $\xi \in \mathrm{S}_n(0)$, let $U_{\xi,N}$ be its neighbourhood consisting of all ζ such that

$$\forall x \in X_N \qquad \|x\|^\xi - 1/N < \|x\|^\zeta < \|x\|^\xi + 1/N.$$

This means in particular that $\|x_i\|^\zeta < \|x_i\|^\xi + 1$ for all $i < n$, and now an easy calculation together with Proposition 1.7 yields that there exists a constant $C(\xi)$ such that for all N, $U_{\xi,N}$ is contained in the ball of radius $C(\xi)/N$ around ξ, which is what we had to show. $\blacksquare_{1.12}$

This already allows us to construct the following useful tool of variable change in a type.

Definition 1.13. Given a linear map $\varphi\colon E(\bar y) \to E(\bar x)$ extending id_E, we define a pull-back map $\varphi^*\colon \mathrm{S}_{\bar x}(E) \to \mathrm{S}_{\bar y}(E)$ by $\|z\|^{\varphi^*\xi} = \|\varphi z\|^\xi$ (for $z \in E(\bar y)$). For $A \subseteq \mathrm{S}_{\bar y}(E)$, we define $\varphi_* A = (\varphi^*)^{-1}(A) \subseteq \mathrm{S}_{\bar x}(E)$ (this will be particularly convenient in the proof of Lemma 2.10).

Of course, φ is entirely determined by the image $\varphi\bar y$. Thus, when the variables $\bar y$ are known from the context, we may write $\xi\restriction_{\varphi\bar y}$ for $\varphi^*\xi$, so

$$\Big\|a + \sum \lambda_i y_i\Big\|^{\xi\restriction_{\bar z}} = \Big\|a + \sum \lambda_i z_i\Big\|^{\xi}.$$

In fact, we shall often use this latter notation with $\bar z = \bar y$.

Lemma 1.14. *For a fixed tuple* $\bar y \in E(\bar x)^m$*, the map* $\xi \mapsto \xi\restriction_{\bar y}$ *is continuous and Lipschitz. If* $\bar y$ *are linearly independent over* E *then this map is also topologically and metrically open. Moreover, the metric openness is "Lipschitz" as well, in the sense that there exists a constant* $C = C(\bar y)$ *such that for all* ξ *and all* $r > 0$ *we have*

$$B(\xi, r)\restriction_{\bar y} \supseteq B(\xi\restriction_{\bar y}, Cr).$$

Proof. Continuity and the Lipschitz condition are easy. We therefore assume that $\bar y$ are linearly independent over E, and we first prove the moreover part. In the special case where $\bar y$ generate $E(\bar x)$ over E, this is since $(\cdot\restriction_{\bar y})^{-1} = \cdot\restriction_{\bar x}\colon \mathrm{S}_{\bar y}(E) \to \mathrm{S}_{\bar x}(E)$ is Lipschitz. In the general case, we may complete $\bar y$ into a basis for $E(\bar x)$ over E, and using the special case above we reduce to the case where $y_i = x_i$ for $i < m$, which is just Lemma 1.9(v).

For topological openness, we proceed as follows. In the case where $E = 0$, this follows from metric openness and Proposition 1.12. Let us consider now the case where E is finite-dimensional. We fix a basis $\bar b$ for E and a corresponding tuple of variables $\bar w$. We may then identify $E(\bar x)$ with $0(\bar w, \bar x)$, and thus $\bar y$ with its image in $0(\bar w, \bar x)$. We already know that $\cdot\restriction_{\bar w,\bar y}\colon \mathrm{S}_{\bar w,\bar x}(0) \to \mathrm{S}_{\bar w,\bar y}(0)$ is open. In addition, we have a commutative diagram

$$\begin{tikzcd} \mathrm{S}_{\bar w,\bar x}(0) \arrow[rr, "\cdot\restriction_{\bar w,\bar y}"] \arrow[dr, "\cdot\restriction_{\bar w}"'] & & \mathrm{S}_{\bar w,\bar y}(0) \arrow[dl, "\cdot\restriction_{\bar w}"] \\ & \mathrm{S}_{\bar w}(0) & \end{tikzcd}$$

and the map $\cdot\restriction_{\bar y}\colon \mathrm{S}_{\bar x}(E) \to \mathrm{S}_{\bar y}(E)$ is homeomorphic to the fibre of the horizontal arrow over $\mathrm{tp}(\bar b) \in \mathrm{S}_{\bar w}(0)$, so it is open as well. The infinite-dimensional case follows from the finite-dimensional one, since any basic open set in $\mathrm{S}_{\bar x}(E)$ can be defined using finitely many parameters in E. $\blacksquare_{1.14}$

We leave it to the reader to check that if $\bar y$ are not linearly independent over E then $\cdot\restriction_{\bar y}$ is not metrically open, and *a fortiori* not topologically so (consider for example $\cdot\restriction_{x,x}\colon \mathrm{S}_1(0) \to \mathrm{S}_2(0)$).

Lemma 1.15. *Let* $U \subseteq \mathrm{S}_n(E)$ *be open and* $r > 0$*. Then* $B(U, r)$ *is open as well.*

Proof. Let $\bar x$ and $\bar y$ be two n-tuples of variables. Let us identify $\mathrm{S}_n(E)$ with $\mathrm{S}_{\bar x}(E)$, and let $W \subseteq \mathrm{S}_{\bar x,\bar y}(E)$ consist of all ξ such that $\|x_i - y_i\|^\xi < r$ for $i < n$. Then W is open, and by Lemma 1.14 so is $V = \big(W \cap (\cdot\restriction_{\bar x})^{-1}(U)\big)\restriction_{\bar y} \subseteq \mathrm{S}_{\bar y}(E)$. Identifying $\mathrm{S}_{\bar y}(E)$ with $\mathrm{S}_n(E)$ as well, $V = B(U, r)$. $\blacksquare_{1.15}$

Lemma 1.16. *Let* E *be a Banach space.*

(i) *A type in* $\mathrm{S}_n(E)$ *is isolated if and only if all its metric neighbourhoods have non empty interior.*
(ii) *The set of isolated types in* $\mathrm{S}_n(E)$ *is metrically closed.*

Proof. The first assertion follows easily from Lemma 1.15, and the second from the first. $\blacksquare_{1.16}$

Another basic operation one can consider on types is the *restriction of parameters* $\mathrm{S}_n(F) \to \mathrm{S}_n(E)$ when $E \subseteq F$.

Lemma 1.17. *Let* $E \subseteq F$ *be an isometric inclusion of Banach spaces. Then the natural type restriction map* $\pi \colon \mathrm{S}_n(F) \to \mathrm{S}_n(E)$ *is continuous, closed, and satisfies* $\pi B(\xi, r) = B(\pi\xi, r)$.

In particular, π *is both topologically and metrically a quotient map.*

Proof. It is clear that π is continuous. To see that it is closed we use Lemma 1.9. Indeed, since closed sets are exactly those which intersect compact sets on compact sets, it will be enough to show that if $K \subseteq \mathrm{S}_n(E)$ is compact then so is $\pi^{-1}K$, which follows from the characterisation of compact sets as closed and bounded. Finally, it is clear that $d(\xi, \zeta) \geq d(\pi\xi, \pi\zeta)$ for $\xi, \zeta \in \mathrm{S}_n(F)$. Conversely, if $\zeta_0 \in \mathrm{S}_n(E)$ then using Fact 1.1, there exists $\zeta \in \pi^{-1}\zeta_0$ with $d(\xi, \zeta) \geq d(\pi\xi, \zeta_0)$, which proves that $\pi B(\xi, r) = B(\pi\xi, r)$. $\blacksquare_{1.17}$

We also obtain the following result, which is somewhat of an aside with respect to he rest of this paper. We shall therefore allow ourselves to be brief, and assume that the reader is familiar with continuous first order logic (see [BU10, BBHU08]), and, for the part regarding Banach spaces as unbounded metric structures, also with unbounded continuous logic (see [Ben08a]).

Lemma 1.18. *Let* T *be an inductive theory, and for* $n \in \mathbf{N}$ *let* $\mathrm{S}^{\mathrm{qf}}_n(T)$ *denote the space of quantifier-free types consistent with* T*, equipped with the natural logic topology. Assume that, first, every two models of* T *amalgamate over a common substructure, and, second, for every* n*, the variable restriction map* $\mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathrm{S}^{\mathrm{qf}}_n(T)$ *is open. Then* T *admits a model completion, namely a companion which eliminates quantifiers.*

(In fact, an approximate amalgamation property for models of T *over a common finitely generated substructure suffices.)*

Proof. Let $\varphi(\bar{x}, y)$ be a quantifier-free formula, inducing a continuous function $\hat{\varphi} \colon \mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathbf{R}$ (which has compact range, by compactness of $\mathrm{S}^{\mathrm{qf}}_{n+1}(T)$). Let $\pi \colon \mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathrm{S}^{\mathrm{qf}}_n(T)$ denote the variable restriction map, and define $\rho \colon \mathrm{S}^{\mathrm{qf}}_n(T) \to \mathbf{R}$ as the infimum over the fibre:

$$\rho(q) = \inf\bigl\{\hat{\varphi}(p) \colon \pi p = q\bigr\}.$$

Since π is continuous (automatically) and open (by hypothesis), ρ is continuous as well, and can therefore be expressed as a uniform limit of $\hat{\psi}_n \colon \mathrm{S}^{\mathrm{qf}}_n(T) \to \mathbf{R}$, where $\psi_n(\bar{x})$ are quantifier-free formulae, say $\|\rho - \hat{\psi}_n\| \leq 2^{-n}$. One can now express that $\sup_{\bar{x}} |\psi_n(\bar{x}) - \inf_y \varphi(\bar{x}, y)| \leq 2^{-n}$ for all n by a set of sentences.

Let T^* consist of T together with all sentences constructed as above, for all possible quantifier-free formulae $\varphi(\bar{x}, y)$. Then, first, every existentially closed model of T is easily checked to be a model of T^* (using our amalgamation hypothesis), so T and T^* are companions. Moreover, by induction on quantifiers, every formula is equivalent modulo T^* to a uniform limit of quantifier-free formulae, so T^* eliminates quantifiers. $\blacksquare_{1.18}$

Proposition 1.19. *Consider Banach spaces either as metric structures in unbounded continuous logic, or as bounded metric structures via their closed unit balls, as explained, say, in* [Ben09]*. Then (in either approach) the theory of the class of Banach spaces is inductive, and admits a model completion* T^* *which is moreover complete and* $\aleph_0$*-categorical.*

When the entire Banach space is viewed as a structure then the types over a subspace are as per Definition 1.2 and Definition 1.3, and if one only considers the unit ball then the space of I*-types over* $E_{\leq 1}$ *is* $\mathrm{S}^{\leq 1}_I(E) = \bigl\{\xi \in \mathrm{S}_I(E) \colon \|x_i\|^\xi \leq 1 \text{ for all } i \in I\bigr\}$.

Proof. Let us consider the theory T of unit balls of Banach spaces. It is clearly inductive, and it is fairly easy to check that the space of quantifier-free I-types over a unit ball $E_{\leq 1}$ is the space $\mathrm{S}^{\leq 1}_I(E)$ defined in the statement. By the moreover part of Lemma 1.9(v), variable restriction $\mathrm{S}^{\leq 1}_{n+1}(E) \to \mathrm{S}^{\leq 1}_n(E)$ is metrically open. For $E = 0$ this implies in particular that $\mathrm{S}^{\leq 1}_{n+1}(0) \to \mathrm{S}^{\leq 1}_n(0)$ is topologically open, but this latter is just $\mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathrm{S}^{\mathrm{qf}}_n(T)$. This, together with Fact 1.1, fulfils the hypotheses of Lemma 1.18.

By quantifier elimination, $\mathrm{S}_n(T^*) = \mathrm{S}_n^{\mathrm{qf}}(T) = \mathrm{S}_n^{\leq 1}(0)$, so in particular, $\mathrm{S}_0(T^*)$ is a singleton, whereby T^* is complete. Finally, T^* is $\aleph_0$-categorical by the Ryll-Nardzewski Theorem (see [BU07]).

The case of Banach spaces as unbounded structures follows via the bi-interpretability of the whole Banach space with its unit ball. $\blacksquare_{1.19}$

2. The Gurarij space

Definition 2.1. We recall from, say, Lusky [Lus76] that a *Gurarij space* is a Banach space $\mathbf{G}$ having the property that for any $\varepsilon > 0$, finite-dimensional Banach space $E \subseteq F$, and isometric embedding $\varphi\colon E \to \mathbf{G}$, there is a linear map $\psi\colon F \to \mathbf{G}$ extending φ such that in addition, for all $x \in F$, $(1-\varepsilon)\|x\| \leq \|\psi x\| \leq (1+\varepsilon)\|x\|$.

Some authors add the requirement that a Gurarij space be separable, bur from our point of view it seems more elegant to consider separability as a separate property.

Lemma 2.2. *Let F be a Banach space. Then the following are equivalent:*

(i) *The space F is a Gurarij space.*
(ii) *For every n, the set of realised types $\mathrm{tp}(\bar a/F)$, as $\bar a$ varies over F^n, is dense in $\mathrm{S}_n(F)$.*
(iii) *Same for $n = 1$.*

Proof. (i) $\Longrightarrow$ (iii). Let $U \subseteq \mathrm{S}_1(F)$ be open and $\xi \in U$. We may assume that U is defined by a finite set of conditions of the form $\big|\|a_i + r_i x\| - 1\big| < \varepsilon$, where $\|a_i + r_i x\|^\xi = 1$. Let $E \subseteq F$ be the subspace generated by the a_i, and let $E' = E + \mathbf{R}x$ be the extension of E generated by the restriction of ξ to E. By hypothesis, there is a linear embedding $\psi\colon E' \to F$ extending the identity such that $(1-\varepsilon)\|y\| < \|\psi y\| < (1+\varepsilon)\|y\|$ for all $y \in E'$, and in particular for $y = a_i + r_i x$, so $\mathrm{tp}(\psi x/F) \in U$.

(iii) $\Longrightarrow$ (ii). We prove by induction on n, the case $n = 0$ being tautologically true. For the induction step, let $\varnothing \neq U \subseteq \mathrm{S}_{\bar x,y}(F)$ be open, and let $V = U{\restriction}_{\bar x} \subseteq \mathrm{S}_{\bar x}(F)$. By Lemma 1.14, V is open, and by the induction hypothesis there are $\bar b \in F^n$ such that $\mathrm{tp}(\bar b/F) \in V$. Now, consider the map $\theta\colon \mathrm{S}_y(F) \to \mathrm{S}_{\bar x,y}(F)$, sending $\mathrm{tp}(a/F) \mapsto \mathrm{tp}(\bar b, a/F)$. It is continuous (in fact, it is a topological embedding), so $\varnothing \neq \theta^{-1}U \subseteq \mathrm{S}_1(F)$ is open. By hypothesis, there is $c \in F$ such that $\mathrm{tp}(c/F) \in \theta^{-1}U$, i.e., such that $\mathrm{tp}(\bar b, c/F) \in U$, as desired.

(ii) $\Longrightarrow$ (i). Let $E \subseteq E'$ be finite-dimensional, with $E \subseteq F$, and let $\varepsilon > 0$. Let $\bar a$ be a basis for E, and let $\bar a, \bar b$ be a basis for E', say $|\bar a| = n$ and $|\bar b| = m$. For $N \in \mathbf{N}$, let $U_N \subseteq \mathrm{S}_m(F)$ be defined by the (finitely many) conditions of the form $\|\sum s_i a_i + \sum r_j x_j\| \in (1-\varepsilon, 1+\varepsilon)$, where s_i and r_j are of the form $\frac{k}{N}$ and $\|\sum s_i a_i + \sum r_j b_j\| \in (1-\varepsilon, 1+\varepsilon)$. By hypothesis there is a tuple $\bar c \in F^m$ such that $\mathrm{tp}(\bar c/F) \in U_N$, and we may define $\psi\colon E' \to F$ being the identity on E and sending $\bar b \mapsto \bar c$. For N big enough, it follows from the construction that if $y \in E'$, $\|y\| = 1$ then $\big|\|\psi y\| - 1\big| < 2\varepsilon$, which is good enough. $\blacksquare_{2.2}$

Model theorists may find the second and third conditions of Lemma 2.2 reminiscent of a topological formulation of the Tarski-Vaught Criterion: a metrically closed subset A of a structure is an elementary substructure if and only if the set of types over A realised in A is dense. Indeed,

Theorem 2.3. *Let T^* be the model completion of the theory of Banach spaces, as per Proposition 1.19. Then its models are exactly the Gurarij spaces. In particular, since T^* is $\aleph_0$-categorical, there exists a unique separable Gurarij space (up to isometric isomorphism).*

Proof. Let E be a Banach space, and embed it in a model $F \vDash T^*$. Then, first, by quantifier elimination, E is a model of T^* if and only if $E \preceq F$. Second, by the topological Tarski-Vaught Criterion evoked above, $E \preceq F$ if and only if the set of types over E, in the sense of $\mathrm{Th}(F) = T^*$, realised in E, is dense.

By Proposition 1.19 the space of types over E (in the sense of $T^* = \mathrm{Th}(E)$) is $\mathrm{S}_n^{\leq 1}(E)$ as defined there. By a dilation argument, the set of types realised in E is dense in $\mathrm{S}_1(E)$ if and only if the set of types realised in $E_{\leq 1}$ is dense in $\mathrm{S}_1^{\leq 1}(E)$, and we conclude by Lemma 2.2 (or, if one works with the whole space as an unbounded structure, the same holds without the dilation argument). $\blacksquare_{2.3}$

As mentioned in the introduction, the isometric uniqueness of the separable Gurarij space was originally proved by Lusky [Lus76] using the Lazar-Lindenstrauss matrix representation of L^1 pre-duals. The same was recently re-proved by Kubiś and Solecki [KS] using more elementary methods. Upon careful reading, their argument essentially consist of showing that the separable Gurarij space is the Fraïssé limit of the class of finite-dimensional Banach spaces, as is pointed out, alongside a general development of Fraïssé theory for metric structures (yielding yet another proof of the same result) by the first author [Bena]. From this point onward we shall leave continuous logic aside, and work entirely within the

formalism of type spaces as introduced in Section 1. As we shall see, the uniqueness and existence also follow as easy corollaries from later results which do not depend explicitly on any form of formal logic (Corollary 2.7 and Lemma 2.10).

Definition 2.4. Let E be a Banach space. We say that a Banach space F is *atomic* over E if $E \subseteq F$ and the type over E of every finite tuple in F is isolated.

By Proposition 1.12, every Banach space is atomic over 0.

Theorem 2.5. *Let $E \subseteq F_0 \subseteq F_1$ be Banach spaces with $\dim F_0/E$ finite and F_1 separable and atomic over E, let $\mathbf{G} \supseteq E$ be a Gurarij space, and let $\varphi\colon F_0 \to \mathbf{G}$ be an isometric embedding extending id_E. Then there exist isometric embeddings $\psi\colon F_1 \to \mathbf{G}$ extending id_E with $\|\psi{\restriction}_{F_0} - \varphi\|$ arbitrarily small.*

In particular, any separable Banach space atomic over E embeds isometrically over E in any Gurarij space containing E.

Proof. It is enough to prove this in the case where $\dim F_1/F_0 = 1$. We may then choose a basis $\bar{a} \in F_1^{n+1}$ for F_1 over E, such that in addition $a_0, \ldots, a_{n-1}$ generate F_0. By hypothesis, $\xi = \mathrm{tp}(\bar{a}/E) \in \mathrm{S}_{n+1}(E)$ is isolated. Let $\rho\colon \mathrm{S}_{n+1}(\mathbf{G}) \to \mathrm{S}_{n+1}(E)$ be the parameter restriction map, and let $K = \rho^{-1}(\xi)$, observing that for any $\varepsilon > 0$, $B(K, \varepsilon) = \rho^{-1}B(\xi, \varepsilon)$ is a neighbourhood of K. We construct a sequence of tuples $\bar{c}_k \in \mathbf{G}^{n+1}$, each of which realising a type in $B(\xi, 2^{-k}r)$, as follows.

For $k = 0$, we let $V \subseteq \mathrm{S}_{n+1}(\mathbf{G})$ be the set defined by $\|x_i - \varphi a_i\| < r$ for $i < n$, which is open and intersects K. Then $V \cap B(K, r)^\circ \neq \varnothing$ (where $\cdot^\circ$ denotes topological interior), and we choose $\bar{c}_0$ to realise some type there. Given $\bar{c}_k$, we let $U_k \subseteq \mathrm{S}_{n+1}(\mathbf{G})$ be the set defined by $\|x_i - c_{k,i}\| < 2^{-k}r$ for $i \leq n$, which is again open intersecting K, and we choose $\bar{c}_{k+1}$ to realise a type in $U_k \cap B(K, 2^{-n-1}r)^\circ$.

We obtain a Cauchy sequence $(\bar{c}_k)$ converging to some $\bar{c} \in \mathbf{G}^{n+1}$, whose type $\mathrm{tp}(\bar{c}/E)$, being the metric limit of $\mathrm{tp}(\bar{c}_k/E)$, must be ξ. Then the linear map $\psi\colon F_1 \to \mathbf{G}$ which extends id_E by $a_i \mapsto c_i$ is an isometric embedding.

Finally, reading through our construction, we have $\|\varphi a_i - c_i\| < 3r$ for all $i < n$, and choosing r small enough, $\|\psi{\restriction}_{F_0} - \varphi\|$ is as small as desired. $\blacksquare_{2.5}$

In particular, any two separable Gurarij spaces atomic over E embed in one another, but we can do better.

Theorem 2.6. *Let $\mathbf{G}_i$ be separable Gurarij spaces atomic over E for $i = 0, 1$, let $E \subseteq F \subseteq \mathbf{G}_0$ with $\dim F/E$ finite, and let $\varphi\colon F \to \mathbf{G}_1$ be an isometric embedding extending id_E. Then there exist isometric isomorphisms $\psi\colon \mathbf{G}_0 \cong \mathbf{G}_1$ extending id_E with $\|\psi{\restriction}_F - \varphi\|$ arbitrarily small.*

In particular, any two separable Gurarij spaces atomic over E are isometrically isomorphic over E.

Proof. Follows from Theorem 2.5 by a back-and-forth argument. $\blacksquare_{2.6}$

Since every Banach space is atomic over 0, we obtain the uniqueness and universality of the separable Gurarij space.

Corollary 2.7. *Every two separable Gurarij spaces are isometrically isomorphic, and every separable Banach space embeds isometrically in any Gurarij space (separable or not).*

Similarly, the Gurarij space is *approximately homogeneous*:

Corollary 2.8. *Let $\mathbf{G}$ be a separable Gurarij space, let $F \subseteq \mathbf{G}$ be finite-dimensional, and let $\varphi\colon F \to \mathbf{G}$ is an isometric embedding. Then there exist isometric automorphisms $\psi \in \mathrm{Aut}(\mathbf{G})$ such that $\|\psi{\restriction}_F - \varphi\|$ is arbitrarily small.*

Moreover, if $E \subseteq F$ is such that $\mathbf{G}$ is atomic over E, and $\varphi{\restriction}_E = \mathrm{id}$, then we may require that $\psi{\restriction}_E = \mathrm{id}$ as well.

Notation 2.9. We shall denote by $\mathbf{G}$ the unique separable Gurarij space. Similarly, for a separable Banach space E, we let $\mathbf{G}[E]$ denote the unique atomic separable Gurarij space over E, if such exists, observing that since all types over 0 are isolated, $\mathbf{G} = \mathbf{G}[0]$.

We now turn to a criterion for the existence of $\mathbf{G}[E]$.

Lemma 2.10. *Let E be a separable Banach space, and say that a type $\xi \in \mathrm{S}_{\mathbf{N}}(E)$ is a* Gurarij type *if it generates a Gurarij space. Then the set of Gurarij types over E is co-meagre in $\mathrm{S}_{\mathbf{N}}(E)$. Moreover, there exists a dense G_δ set $Z \subseteq \mathrm{S}_{\mathbf{N}}(E)$ such that if some $\xi \in Z$ generates F then F is Gurarij and $\{x_i\}_{i \in \mathbf{N}} \subseteq F$ is dense.*

In particular, the separable Gurarij space $\mathbf{G}$ exists.

Proof. Let $X = \{x_i\}_{i\in\mathbf{N}}$, so $\mathrm{S}_X(E) = \mathrm{S}_{\mathbf{N}}(E)$. Let y be a new variable symbol. For $k \in \mathbf{N}$, let $[\varnothing \to y]\colon E(X) \to E(X,y)$ denote $\mathrm{id}_{E(X)}$, let $[x_k \to y]\colon E(X) \to E(X,y)$ be defined as $\mathrm{id}_{E(X\smallsetminus\{x_k\})}$ together with $x_k \mapsto y$, and let $[y \to x_k]\colon E(X,y) \to E(X)$ be defined as $\mathrm{id}_{E(X)}$ together with $y \mapsto x_k$.

The space $\mathrm{S}_{X,y}(E)$ has a countable base of open sets $\{U_n\}_{n\in\mathbf{N}}$, and we may assume furthermore that each $U_n \neq \varnothing$ can be defined using only variables from among $\{x_0,\ldots,x_{n-1},y\}$, so $[y \to x_n]_*U_n = [x_n \to y]^*U_n \neq \varnothing$. For each n we define $Z_n \subseteq \mathrm{S}_X(E) = \mathrm{S}_{\mathbf{N}}(E)$ to consist of all types ξ such that either

- $\xi \notin [\varnothing \to y]^*U_n$, which defines a closed set, since $[\varnothing \to y]^*$ is open, or
- there exists some k such that $\xi \in [y \to x_k]_*U_n$, which defines an open set.

Then Z_n is a G_δ set, and we claim that it is dense. Indeed, let $\varnothing \neq W \subseteq \mathrm{S}_X(E)$ be open. We may assume that $U_n \cap [\varnothing \to y]_*W \neq \varnothing$, since otherwise $W \cap Z_n \neq \varnothing$. Then there exist k such that $U_k \subseteq U_n \cap [\varnothing \to y]_*W$, so

$$\begin{aligned}\varnothing \neq [y \to x_k]_*U_k &\subseteq [y \to x_k]_*U_n \cap [y \to x_k]_*[\varnothing \to y]_*W \\ &= [y \to x_k]_*U_n \cap W \\ &\subseteq Z_n \cap W,\end{aligned}$$

proving our claim.

Now let $Z = \bigcap Z_n$, a dense G_δ set, and we claim that every $\xi \in Z$ is Gurarij. Indeed, let ξ generate F, and let $\varnothing \neq U \subseteq \mathrm{S}_1(F)$ be open. We define $\theta\colon \mathrm{S}_1(F) \to \mathrm{S}_{X,y}(E)$ as in the proof of Lemma 1.14 (working over E, whereas there we worked over 0), and there exists n such that $U \supseteq \theta^{-1}(U_n) \neq \varnothing$. Since $\theta^{-1}(U_n) \neq \varnothing$, we have $\xi \in [\varnothing \to y]^*U_n \cap Z_n$, and so for some k we have $[y \to x_k]^*\xi \in U_n$. This means exactly that $\mathrm{tp}(x_k/F) \in \theta^{-1}(U_n) \subseteq U$, showing that ξ is indeed Gurarij. Moreover, we have shown that every open set $\varnothing \neq U \subseteq \mathrm{S}_1(F)$ is realised in F by some x_i, from which it follows that $\{x_i\}_{i\in\mathbf{N}}$ is dense in F. $\blacksquare_{2.10}$

Notice that since a Banach space has no isolated points, if a sequence is dense there then every tail of the sequence is dense there as well.

Theorem 2.11 (Omitting Types Theorem for Gurarij spaces)**.** *Let E be a separable Banach space, and for each n, let $X_n \subseteq \mathrm{S}_n(E)$ be metrically open and topologically meagre. Then there exists a separable Gurarij space $\mathbf{G} \supseteq E$ such that in addition, for every n, no type in X_n is realised in $\mathbf{G}$ (we then say that $\mathbf{G}$* omits *all X_n). Moreover, the set of Gurarij types which generate such spaces is co-meagre.*

Proof. Let $Z \subseteq \mathrm{S}_{\mathbf{N}}(E)$ be the set produced by Lemma 2.10. For each n, let $[\mathbf{N}]^n = \{s \subseteq \mathbf{N}\colon |s| = n\}$. For $s \in [\mathbf{N}]^n$ can be enumerated uniquely as an increasing sequence $\{k_0,\ldots,k_{n-1}\}$, and we then define $[s]\colon E(n) \to E(\mathbf{N})$ by $x_i \mapsto x_{k_i}$ for $i < n$. Then $[s]^*\colon \mathrm{S}_{\mathbf{N}}(E) \to \mathrm{S}_n(E)$ is continuous and open, so $[s]_*X_n \subseteq \mathrm{S}_{\mathbf{N}}(E)$ is meagre. Since everything is countable,

$$Z_1 = Z \smallsetminus \bigcap_{n,s\in[\mathbf{N}]^n} [s]_*X_n$$

is co-meagre as well. All we need to show is that if $\xi \in Z_1$ generates $\mathbf{G}$ then $\mathbf{G}$ omits X_n. Indeed, assume that some $\xi \in X_n$ is realised in $\mathbf{G}$, say by $\bar{a}$. Since X_n is metrically open, there exists $r > 0$ such that $B(\xi,r) \subseteq X_n$. Since the sequence $\{x_i\}$ is dense in $\mathbf{G}$, there is an increasing sequence $k_0 < \ldots < k_{n-1}$ such that $\|x_{k_j} - a_j\| < r$. But then $\mathrm{tp}(x_{\bar{k}}/E) \in X_n$, so $\xi \in [\bar{k}]_*X_n$, contradicting the choice of ξ and completing the proof. $\blacksquare_{2.11}$

Corollary 2.12 (Criterion for primeness over E)**.** *Let G be a Gurarij space, and let $E \subseteq G$ be a separable subspace. Then the following are equivalent:*

(i) *The space G is* prime over E*, that is to say that it embeds isometrically over E in every Gurarij space containing an isometric copy of E.*
(ii) *The space G is separable and atomic over E, namely, $G = \mathbf{G}[E]$.*

Proof. It is immediate from Theorem 2.5 that $\mathbf{G}[E]$ is prime over E. For the other direction, assume that $\mathbf{G}$ is prime over E. Since E is separable, it embeds (by Theorem 2.5) in a separable Gurarij space, so $\mathbf{G}$ must be separable as well. Finally, assume toward a contradiction that $\mathbf{G}$ realises some non isolated type ξ. By Lemma 1.16 there exists $r > 0$ such that the closed metric ball $\overline{B}(\xi,r)$ has empty interior. Since the metric is lower semi-continuous, the closed metric ball is topologically closed, and is therefore meagre, as is the open ball $B(\xi,r)$. By Theorem 2.11, there exists a separable Gurarij space $\mathbf{G} \supseteq E$ which omits $B(\xi,r)$. Thus G cannot embed over E in $\mathbf{G}$, a contradiction. $\blacksquare_{2.12}$

Proposition 2.13. *Let E be a separable Banach space. Then $\mathbf{G}[E]$ exists if and only if, for each n, the set of isolated types in $\mathrm{S}_n(E)$ is dense.*

Proof. Assume first that the sets of isolated types are dense. For a given n, let $\mathscr{I}_n$ be the set of isolated types in $\mathrm{S}_n(E)$, and assume that it is dense. Then $B(\mathscr{I}_n, r)$ contains a dense open set, and $\bigcap_{r>0} B(\mathscr{I}_n, r) = \overline{\mathscr{I}}_n$ is co-meagre. By Lemma 1.16 we have $\mathscr{I}_n = \overline{\mathscr{I}}_n$, so $\mathrm{S}_n(E) \smallsetminus \mathscr{I}_n$ is meagre and metrically open. By Theorem 2.11, if $\mathscr{I}_n$ is dense for all n then an atomic separable Gurarij space over E exists.

Conversely, assume that $\mathbf{G}[E]$ exists. Then the set of n-types over E realised in $\mathbf{G}[E]$ is dense (by Lemma 2.2), and they are all isolated. $\blacksquare_{2.13}$

Model theorists will recognise Proposition 2.13 as the usual criterion for the existence of an atomic model, and as such it is in no way particular to Banach spaces. In the specific context of Banach spaces, however, it can be improves as follows.

Lemma 2.14. *For a type $\xi \in \mathrm{S}_{\bar{x}}(E)$ the following are equivalent*

(i) *The type ξ is isolated.*
(ii) *The type $\xi\restriction_{\bar{y}}$ is isolated for $\bar{y} \in E(\bar{x})^m$ (and every m).*
(iii) *The type $\xi\restriction_y$ is isolated for every $y \in E(\bar{x})$.*

Proof. (i) $\Longrightarrow$ (ii). When $\bar{y}$ are linearly independent over E, this follows from Lemma 1.14. For the general case, it will now be enough to consider the case where $\bar{y}$, of length m, extends the original tuple of variables $\bar{x}$, and for $j < m$ let us write $y_j = a_j + \sum \lambda_{ij} x_i$. Given $r > 0$, there exists by hypothesis an open set U such that $\xi \in U \subseteq B(\xi, r)$, and let $V = (\cdot\restriction_{\bar{x}})^{-1} U \subseteq \mathrm{S}_{\bar{y}}(E)$. Intersecting V with the open sets defined by $\|y_j - \sum_{i<m} \lambda_{ij} y_i\| < r$ we obtain an open set V' with $\xi\restriction_{\bar{y}} \to V' \subseteq B(\xi\restriction_{\bar{y}}, r')$ for some $r' = r'(r, \bar{y})$ which goes to zero with r.

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). We repeat the proof of Proposition 1.12 (in fact, that result is merely a special case of the present, alongside the fact that types in $\mathrm{S}_1(0)$ are trivially isolated). Indeed, for each N there exists by hypothesis a neighbourhood $U_N \ni \xi$ consisting of ζ such that

$$\forall y \in X_N \qquad d(\zeta\restriction_y, \xi\restriction_y) < 1/N.$$

Using Proposition 1.7 we conclude as for Proposition 1.12. $\blacksquare_{2.14}$

Theorem 2.15. *The following are equivalent for a separable Banach space E:*

(i) *The space $\mathbf{G}[E]$ exists.*
(ii) *For each n, the set of isolated types in $\mathrm{S}_n(E)$ is dense.*
(iii) *The set of isolated types in $\mathrm{S}_1(E)$ is dense.*

Proof. We only need to show that if the set of isolated 1-types is dense then $\mathbf{G}[E]$ exists. Indeed, proceeding as in the proof of Proposition 2.13 there exists a separable Gurarij space $\mathbf{G} \supseteq E$ which only realises isolated 1-types over E. By Lemma 2.14, $\mathbf{G}$ is atomic over E. $\blacksquare_{2.15}$

Corollary 2.16. *Let E be a separable Banach space, and let $H = \mathrm{Aut}(\mathbf{G})$ act by composition on the space of linear isometric embeddings $X = \mathrm{Emb}(E, \mathbf{G})$, where both are equipped with the topology of point-wise convergence (the strong operator topology).*

(i) *The space X is Polish, the action $H \curvearrowright X$ is continuous and all its orbits are dense.*
(ii) *If $\mathbf{G}[E]$ exists, then the set of $\varphi \in X$ such that $\mathbf{G}$ is atomic over φE (call these* atomic embeddings*) is a dense G_δ orbit under this action.*
(iii) *If $\mathbf{G}[E]$ does not exist then there are no atomic embeddings and all orbits are meagre.*

Proof. The first item is easy and left to the reader (density is by Corollary 2.8).

It follows from Theorem 2.6 that the set $Z \subseteq \mathrm{Emb}(E, \mathbf{G})$ of atomic embeddings forms a single orbit under $\mathrm{Aut}(\mathbf{G})$. By definition, $Z \neq \varnothing$ if and only if $\mathbf{G}[E]$ exists. Let $\mathscr{I}_n \subseteq \mathrm{S}_n(E)$ denote the set of isolated types. For $r > 0$, we know that $B(\mathscr{I}_n, r)$ is a neighbourhood of $\mathscr{I}_n$, so there exists an open set $U_{n,r}$ such that $\mathscr{I}_n \subseteq U_{n,r} \subseteq B(\mathscr{I}_n, r)$ (in fact one can show that $B(\mathscr{I}_n, r)$ is open, but we shall not require this). For each $\bar{b} \in \mathbf{G}^n$, we define $V_{\bar{b},r} \subseteq \mathrm{Emb}(E, \mathbf{G})$ to consist of all φ such that $\mathrm{tp}(\bar{b}/\varphi E) \in \varphi U_{n,r}$. It is easy to see that since $U_{n,r}$ is open, so is $V_{\bar{b},r}$. Since the set of isolated types is metrically closed, we have

$$Z = \bigcap_{n, \bar{b} \in \mathbf{G}^n, r>0} V_{\bar{b},r} = \bigcap_{n, \bar{b} \in \mathbf{G}_0^n, k} V_{\bar{b},2^{-k}},$$

where $\mathbf{G}_0 \subseteq \mathbf{G}$ is any countable dense subset. Thus, if $Z \neq \varnothing$ it is a dense G_δ orbit.

Assume now that $\mathbf{G}[E]$ does not exist, namely, that isolated types are not dense, and let $\psi \in X$. Then $\mathbf{G}$ necessarily realises some type $\psi\xi \in \mathrm{S}_n(\psi E)$ where $\xi \in \mathrm{S}_n(E)$ is non isolated. By Lemma 1.16, for $r > 0$ small enough, the closed metric ball $\overline{B}(\xi, r)$ is (topologically) closed of empty interior. For $\bar{b} \in \mathbf{G}^n$, let $V_{\bar{b}} \subseteq \mathrm{Emb}(E, \mathbf{G})$ consist of all φ such that $\mathrm{tp}(\bar{b}/\varphi E) \notin \overline{B}(\varphi\xi, r)$. Reasoning as above, each $V_{\bar{b}}$ is a dense open set, and the set of $\varphi \in X$ such that $\mathbf{G}$ omits $\varphi\xi$ is co-meagre. Since this set is also disjoint from the orbit of ψ, we are done. $\blacksquare_{2.16}$

3. Isolated types over one-dimensional spaces

In this section we shall attempt to characterise isolated types over arbitrary finite-dimensional E. We start with the next-easiest case after $E = 0$, namely when $\dim E = 1$. Even though this case will be fully subsumed in the general finite-dimensional case, it is technically significantly simpler and deserves some specific comments, so we chose to treat it separately.

Definition 3.1. A *norming linear functional* for $v \in E \setminus \{0\}$ is a continuous linear functional $\lambda \in E^*_{=1}$ such that $\lambda v = \|v\|$.

By the Hahn-Banach Theorem, a norming linear functional always exists. We say that v is *smooth* in E if the norming linear functional is unique.

Proposition 3.2. *Let E be a Banach space, and let $v \in E \setminus \{0\}$. Then E is atomic over v if and only if v is smooth in E.*

Proof. By Lemma 2.14, we may assume that $E = \langle v, u\rangle$ and show that $\mathrm{tp}(u/v)$ is isolated if and only if v is smooth in E. Assume first that for some $s, \varepsilon > 0$ and $D \in \mathbf{R}$ we have

$$\|v \pm su\| < \|v\| \pm sD + s\varepsilon.$$

It follows by the triangle inequality that

$$\|v\| \pm tD - t\varepsilon \leq \|v \pm tu\| < \|v\| \pm tD + t\varepsilon, \qquad 0 < t \leq s,$$

or equivalently,

$$\big|\| \pm rv + u\| - r\|v\| \mp D\big| < \varepsilon, \qquad r \geq s^{-1}.$$

If v is smooth, let λ be the unique norming functional, and let $D = \lambda u$. Then for any $\varepsilon > 0$ there exists s as above. Then $\xi = \mathrm{tp}(u/v)$ satisfies the open condition $\|v \pm sx\| < \|v\| \pm sD + s\varepsilon$, which in turn implies that $\big|\|rv - u\| - \|rv - x\|\big| \leq 2\varepsilon$ for all $|r| \geq s^{-1}$. Finitely many additional open condition can ensure that that the same holds for all r yielding an open set $\xi \in U \subseteq B(\xi, 3\varepsilon)$, showing that ξ is isolated.

Conversely, if v is not smooth then there are norming functionals $\lambda^{\pm}$, where $D^- = \lambda^- u < D^+ = \lambda^+ u$. Any neighbourhood of ξ contains one U which is defined by finitely many conditions of the form $\big|\|r_i v + x\| - \|r_i v + u\|\big| < \varepsilon$. We can construct a Banach space E' generated by $\{v, w\}$, with $\|v\|$ as in E, such that $\zeta = \mathrm{tp}(w/v) \in U$ and v is smooth in E', with unique norming functional being defined by $\mu w = D^-$. This means that for r big enough we have

$$\|rv + w\| \approx r\|v\| + D^- \leq \|rv + u\| + D^- - D^+,$$

so $d(\xi, \zeta) \geq D^+ - D^-$. Therefore $B(\xi, D^+ - D^-)$ is *not* a topological neighbourhood of ξ, and ξ is not isolated. $\blacksquare_{3.2}$

We provided a fairly elementary argument to the "only if" part of Proposition 3.2. The machinery developed above provides us with a conceptually different argument, which in a sense we find preferable. First, let us recall that by Mazur [Maz33, Satz 2], the set of smooth points in the unit sphere of a separable Banach space is a dense G_δ. Assume now that E is atomic over v, and without loss of generality, say that $\|v\| = 1$, and let $u \in \mathbf{G}$ be smooth of norm one. By Theorem 2.5 there exists an isometric embedding of E in $\mathbf{G}$ sending v to u, so v must be smooth.

Yet a third way to prove that if E is atomic over v then v is smooth in E is via Lemma 1.11. We follow this path in a more general case below, see Theorem 5.7.

It follows from Lemma 1.17 that if $E \subseteq F$, and the topology and metric coincided on $\mathrm{S}_n(F)$, then they would also coincide on $\mathrm{S}_n(E)$, which would mean that every type in $\mathrm{S}_n(E)$ is isolated. Given Proposition 3.2, it follows that the metric strictly refines the topology on $\mathrm{S}_n(E)$ for every $E \neq 0$.

We obtain the following result, stated by Lusky [Lus76] (the proof is not spelled out explicitly, but given as "apply the following modifications to the proof of the uniqueness of the Gurarij space").

Corollary 3.3. *The smooth points in the unit sphere of* $\mathbf{G}$ *form a single dense* G_δ *orbit under isometric automorphisms.*

Proof. Immediate from Proposition 3.2 and Theorem 2.6. $\blacksquare_{3.3}$

4. The Legendre-Fenchel transformation of 1-types

We shall eventually extend Proposition 3.2 to types over an arbitrary finite-dimensional space E. We start with a few technical preliminaries, which hold whether $\dim E$ is finite or infinite. The notion of smoothness of a single vector, or of the one-dimensional vector space it generates, will be replaced in dimension higher than one with the following generalisation (see Phelps [Phe60]).

Definition 4.1. We say that an extension of Banach space $E \subseteq F$ has *property* U, or that E *has property* U *in* F, if every continuous linear functional on E has a unique extension to F of the same norm.

We say that a type $\xi \in \mathrm{S}_I(E)$ has *property* U if E has property U in $E[\xi]$.

We say that a 1-type $\xi \in \mathrm{S}_1(E)$ has *property* U_ε, for some $\varepsilon > 0$, if for every $\lambda \in E^*_{=1}$ and two norm-preserving extensions λ', λ'' to $E[\xi]$, we have $|\lambda' x - \lambda'' x| < \varepsilon$.

If $E \subseteq F$ and $\dim E = 1$ then E has property U in F if and only if one (every) non trivial vector in E is smooth in F. Also, a 1-type has property U if and only if it has property U_ε for every $\varepsilon > 0$.

Lemma 4.2. *Let* E *be a Banach space. Then the set of types in* $\mathrm{S}_n(E)$ *with property* U *is metrically closed.*

Proof. Assume that $\xi_k \to \xi$ in d, where each ξ_k has property U. Possibly passing to a sub-sequence we may assume that $d(\xi_k, \xi_{k+1}) < 2^{-k}$, and using our basic amalgamation result Fact 1.1 we can construct an extension $F \supseteq E$ containing realisations $\bar v_k$ of ξ_k, such that in addition $d(\bar v_k, \bar v_{k+1}) < 2^{-k}$, and which is moreover generated as a Banach space by these realisations. Then $\bar v = \lim \bar v_k$ realises ξ and E has property U in F, so *a fortiori* in $E[\xi]$. $\blacksquare_{4.2}$

Lemma 4.3. *An arbitrary extension* $E \subseteq F$ *has property* U *if and only if* $\mathrm{tp}(x/E)$ *has it for every* $x \in F$. *In other words, Lemma 2.14 holds with isolation replaced with property* U.

Proof. Immediate. $\blacksquare_{4.3}$

We have thus reduced both isolated and property U to 1-types. This will be most useful in conjunction with the following alternative characterisation of 1-types over a normed space E, introduced in [Benb] (see also Uspenskij [Usp08]).

Definition 4.4. Let X be an arbitrary metric space. A *Katětov function* on X is a function $f\colon X \to \mathbf{R}$ satisfying $f(x) \leq f(y) + d(x, y)$ and $d(x, y) \leq f(x) + f(y)$ for all $x, y \in X$. The space of Katětov functions on X is denoted $K(X)$. As with type spaces, we equip $K(X)$ with a double structure, the topology of point-wise convergence and the metric of uniform convergence (i.e., the supremum metric).

If X is a normed space, or a convex subset thereof, we let $K_C(X)$ denote the space of *convex* Katětov functions on X, with the induced topometric structure.

Fact 4.5. *Let* ξ *be a* 1*-type over* E, *and let* $f_\xi(a) = \|x - a\|^\xi$ *for* $a \in E$. *Then*

(i) *The map* $\xi \mapsto f_\xi$ *defines a bijection between* $S_1(E)$ *and* $K_C(E)$, *whose inverse is given by*

$$\|\alpha x - a\|^\xi = \begin{cases} \|a\| & \alpha = 0 \\ |\alpha| \xi(a/\alpha) & \alpha \neq 0. \end{cases}$$

(ii) *This bijection is a topological homeomorphism and a metric isometry.*

Proof. The first item is [Benb, Lemma 1.2]. For the second, that the bijection is homeomorphic (in the respective topologies of point-wise convergence) follows easily from the characterisation of the inverse, while the isometry is exactly Proposition 1.7 for 1-types. $\blacksquare_{4.5}$

Consequently, from now on we shall identify $K_C(E)$ with $\mathrm{S}_1(E)$.

Fact 4.6. *Let* $X \subseteq Y$ *be metric spaces, and for* $f \in K(X)$ *and* $y \in Y$ *define*

$$\tilde f(y) = \inf_{x \in X} f(x) + d(x, y).$$

Then $\tilde f \in K(Y)$ *extends* f, *and the induced embedding* $K(X) \subseteq K(Y)$ *is isometric. When* $Y = E$ *is a normed space,* $X \subseteq E$ *is convex and* $f \in K_C(X)$, *the extension* $\tilde f$ *is convex as well, inducing an isometric embedding* $K_C(X) \subseteq K_C(E)$.

Proof. The first assertion goes back to Katětov [Kat88], and the second is [Benb, Lemma 1.3(i)]. $\blacksquare_{4.6}$

Question 4.7. If $X \subseteq E$ is convex and compact (or totally bounded) then the topology and metric agree on $K_C(X)$, and it follows that the inclusion $K_C(X) \subseteq K_C(E)$ is also continuous, and therefore homeomorphic (since the restriction map is always continuous). At the other extremity, if $X = E$ then the inclusion is homeomorphic as well. What about the general case?

A useful tool in analysing convex Katětov functions (or convex functions in general) is *convex conjugation*, or the *Legendre-Fenchel transformation.* In the infinite-dimensional case we consider E and E^* with their norm typologies, unless explicitly said otherwise.

Fact 4.8. *Let $f \colon E \to \mathbf{R} \cup \{\infty\}$ be convex and lower semi-continuous, not identically ∞ (what Rockafellar calls a* closed proper convex function*). Call* $\operatorname{dom} f = \{v \in E : f(v) < \infty\}$*, the* domain *of f, itself a non empty convex set, and define the* conjugate $f^* \colon E^* \to \mathbf{R} \cup \{\infty\}$ *by*

$$f^*(\lambda) = \sup_{v \in E} \lambda v - f(v) = \sup_{v \in \operatorname{dom} f} \lambda v - f(v) \in \mathbf{R} \cup \{\infty\}.$$

Then f^* is again convex, lower semi-continuous in the weak topology, and a fortiori in the norm topology, not identically ∞, and $f = f^{**}\restriction_E$ under the canonical identification $E \subseteq E^{**}$. In particular, f is lower semi-continuous in the weak topology.*

Moreover, if g is another such function, then $\|f - g\| = \|f^* - g^*\|$, where $\|\cdot\|$ denotes the supremum norm, possibly infinite, and we agree that $|\infty - \infty| = 0$.

Proof. For the finite-dimensional case, see Rockafellar [Roc70, Section 12]. The general case is proved essentially in the same fashion, using the version of the Hahn-Banach Theorem stating that a closed convex set is the intersection of the closed half-spaces which contain it.

The moreover part is easy to check directly. $\blacksquare_{4.8}$

Clearly, $\lambda v \leq f(v) + f^*(\lambda)$ for any v, λ. The following is an easy criterion which would allow us to obtain equality.

Corollary 4.9. *Let f be as in Fact 4.8, let $R \geq 0$, and let $v \in \operatorname{dom} f$ be such that f is (finite and) R-Lipschitz in some neighbourhood of v. Then there exists $\lambda \in E^*$ such that $\lambda v = f(v) + f^*(\lambda)$, and any such λ satisfies $\|\lambda\| \leq R$.*

Proof. Let r be such that f is R-Lipschitz on $B(v, r)$, and for $t > 0$ let $S_t \subseteq E^*$ be the set of λ such that $\lambda v \geq f(v) + f^*(\lambda) - t$. This set is non empty (since $f^{**}(v) = f(v)$ is finite) and weak*-closed (since f^* is lower semi-weak*-continuous). Let $\lambda \in S_t$, and let $u \in E_{=1}$ norm λ. Then

$$f(v) + rR \geq f(v + ru) \geq \lambda(v + ru) - f^*(\lambda) \geq f(v) + r\|\lambda\| - t.$$

Thus $\|\lambda\| \leq R + \frac{t}{r}$, so S_t is moreover bounded. It follows that $\bigcap_{t>0} S_t$ is non empty and contained in $E^*_{\leq R}$, as desired. $\blacksquare_{4.9}$

Lemma 4.10. *Let $f, g \colon E \to \mathbf{R}$ be lower semi-continuous and convex, let $X \subseteq E$ be convex and open, and let $Y \subseteq E^*$ be the set of λ for which there exists $v \in X$ with $f(v) + f^*(\lambda) = \lambda v$. If g agrees with f on X, then g^* agrees with f^* on Y.*

Proof. Let $\lambda \in Y$, and let $v \in X$ be such that $f(v) + f^*(\lambda) = \lambda v$. Then $f \geq \lambda - \lambda v + f(v)$, and therefore $g\restriction_X \geq \lambda - \lambda v + g(v)$. Since X is open and g convex, this implies $g \geq \lambda - \lambda v + g(v)$ throughout. Then $g^*(\lambda) = \lambda v - g(v) = \lambda v - f(v) = f^*(v)$, as desired. $\blacksquare_{4.10}$

A convex lower semi-continuous function $f \colon E \to \mathbf{R} \cup \{\infty\}$ is essentially the same thing as a convex lower semi-continuous function $f \colon X \to \mathbf{R}$, with convex domain X, such that $\liminf_{v \to u} f(v) = \infty$ for all $u \in \overline{X} \setminus X$. Indeed, we can get one from the other by restricting to the finite domain in one direction, or by extending by ∞ in the other. A special case of the second form is when $X \subseteq E$ is closed and convex and $f \in K_C(X)$. If X is merely convex, every $f \in K_C(X)$, being 1-Lipschitz, admits a unique extension to $\overline{f} \in K_C(\overline{X})$, so requiring X to be closed is not truly a constraint.

Lemma 4.11. *Let $X \subseteq E$ be closed and convex and let $f \in K_C(X)$. Then*

(i) *The domain* $\operatorname{dom} f^*$ *contains the closed unit ball of E^*, and if $\lambda \in \operatorname{dom} f^*$, $\|\lambda\| > 1$, then $f^*(\lambda) = \sup_{v \in \partial X} \lambda v - f(v)$. In particular, if $X = E$ (and $\partial X = \varnothing$) then* $\operatorname{dom} f^*$ *is exactly the closed unit ball.*

(ii) *If X is and $\|\lambda\| = 1$ then $f^*(\lambda) = \sup_{v \in \partial X} \lambda v - f(v)$.*

(iii) *Let $\tilde{f} \in K_C(E)$ be as per Fact 4.6. Then*

$$\tilde{f}^*(\lambda) = \begin{cases} f^*(\lambda) & \|\lambda\| \leq 1 \\ \infty & \|\lambda\| > 1, \end{cases}$$

and conversely,

$$\tilde{f}(v) = \sup_{\|\lambda\| \leq 1} \lambda v - f^*(\lambda).$$

In addition, for $v \notin X$, this is the same as $\sup_{\|\lambda\|=1} \lambda v - f^*(\lambda)$.

(iv) *For $f \in K_C(E)$ and $\lambda \in E^*_{=1}$, the least possible value of a norm-preserving extension of λ at a realisation of $f \in \mathrm{S}_1(E)$ is $f^*(\lambda)$.*

(v) *Let now $f \colon E \to \mathbf{R} \cup \{\infty\}$ be any convex and lower semi-continuous function not identically ∞. Then $f \in K_C(E)$ if and only if $\operatorname{dom} f^* = E^*_{\leq 1}$ and $f^*(\lambda) + f^*(-\lambda) \leq 0$ for all $\lambda \in E^*_{\leq 1}$, or, equivalently, for all $\lambda \in E^*_{=1}$.*

Proof. For the first item, let first $\|\lambda\| \leq 1$, and let $u \in X$ be fixed. Then for all $v \in X$ we have $f(u) \geq \|v-u\| - f(v) \geq \lambda v - f(v) - \|u\|$, whereby $f^*(\lambda) \leq f(u) + \|u\| < \infty$. Now let $\|\lambda\| > 1$ and assume that $\lambda \in \operatorname{dom} f^*$. Choose $u \in E$ which is normed by λ. Then for each $v \in \operatorname{dom} f$, the set $\{v + \alpha u \colon \alpha \geq 0\}$ cannot be contained in X and therefore intersects the boundary, so the supremum is attained on ∂X. A similar consideration also yields the second item.

For the third item, we already know that $\operatorname{dom} \tilde{f}^*$ is exactly the closed unit ball. In addition, it is clear that $\tilde{f}^* \geq f^*$, and if $\|\lambda\| \leq 1$ then, for every $v \in E$ and $u \in X$:

$$\lambda v - \tilde{f}(v) = \lambda v - \inf_{u \in X} \big[f(u) + \|v - u\| \big] \leq \sup_{u \in X} \lambda u - f(u) = f^*(\lambda),$$

whereby $\tilde{f}^*(\lambda) = f^*(\lambda)$. This gives us the first identity, and then Fact 4.8 gives the second one. Now assume that $v \notin X$, so by the Hahn-Banach Theorem there exists $\mu \in E^*_{=1}$ such that $\mu\restriction_X < \mu v$. For any $\lambda \in E^*_{\leq 1}$ there exists $\alpha \geq 0$ such that $\|\lambda + \alpha\mu\| = 1$. Then $f^*(\lambda + \alpha\mu) \leq f^*(\lambda) + \alpha\mu v$, or equivalently, $\lambda v - f^*(\lambda) \leq (\lambda + \alpha\mu)v - f^*(\lambda + \alpha\mu)$, whence it follows that $\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda)$.

The fourth item is immediate. For the fifth, we have already seen that if $f \in K_C(E)$ then $\operatorname{dom} f^* = E^*_{\leq 1}$, and the previous item implies that $f^*(\lambda) + f^*(-\lambda) \leq 0$ for $\lambda \in E^*_{=1}$, and therefore, by convexity, for $\lambda \in E^*_{\leq 1}$. Conversely, assume that $\operatorname{dom} f^* = E^*_{\leq 1}$. Then $f = f^{**}$ is necessarily 1-Lipschitz, and in particular never ∞. Finally, for distinct $v, u \in E$, let $\lambda \in E^*_{=1}$ norm $v - u$. Then

$$f(v) + f(u) \geq \lambda v - f^*(\lambda) - \lambda u - f^*(-\lambda) \geq \lambda(v - u) = \|v - u\|,$$

as desired. $\blacksquare_{4.11}$

Given a closed convex $X \subseteq E$, we shall say that $f \in K_C(X)$ has property U_ε if $\tilde{f}$ does, as a 1-type. By Lemma 4.11, f has property U_ε if and only if $f^*(\lambda) + f^*(-\lambda) > -\varepsilon$ for all $\lambda \in E^*_{=1}$.

Lemma 4.12. *Let $X \subseteq E$ be closed and convex, and let $f \in K_C(X)$, $g \in K_C(E)$ be such that f has property U_ε and $g \leq f$ (i.e., $g \leq \tilde{f}$, or equivalently, $g\restriction_X \leq f$). Then outside X we have $g \geq \tilde{f} - \varepsilon$.*

Proof. Let $v \in E \smallsetminus X$, and assume that $g(v) + \varepsilon < \tilde{f}(v)$. Then there exists $\lambda \in E^*_{=1}$ such that $0 < \lambda v - f^*(\lambda) - g(v) - \varepsilon$. By property U_ε, we get $0 < \lambda v - g(v) + f^*(-\lambda) \leq g^*(\lambda) + g^*(-\lambda)$, which is impossible. $\blacksquare_{4.12}$

5. Isolated types over finite-dimensional spaces

From this point onward, we need to introduce the extra hypothesis that $\dim E < \infty$.

Lemma 5.1. *Let E be a finite-dimensional normed space and let $f \in K_C(E)$ have property U_ε. Then f has property $U_{\varepsilon'}$ for some $\varepsilon' < \varepsilon$, and it admits an open neighbourhood $f \in W \subseteq K_C(E)$ such that every member of W has property U_ε, and $\operatorname{diam}(W) < \varepsilon$.*

In particular, if f has property U then it is isolated.

Proof. For $\lambda \in E^*_{=1}$ choose $\delta_\lambda > 0$ such that $f^*(\lambda) + f^*(-\lambda) + \varepsilon > 4\delta_\lambda$ (ensuring that $\delta_\lambda = \delta_{-\lambda}$), as well as $a_\lambda \in E$ such that $\lambda a_\lambda - f(a_\lambda) + \delta_\lambda > f^*(\lambda)$. Let V_λ be the open ball of radius $\frac{\delta_\lambda}{\|a_\lambda - a_{-\lambda}\|}$ around λ. Since the dimension is finite, $E^*_{=1}$ is compact, and for some finite set $\{\lambda_i\}_{i<n} \subseteq E^*_{=1}$ we have $E^*_{=1} \subseteq \bigcup_i V_{\lambda_i}$. Let $\delta = \min_i \delta_{\lambda_i}$, so f has property $U_{\varepsilon - 3\delta}$, proving our first assertion.

Let $X \subseteq E$ be the convex hull of $\{a_{\lambda_i}\}_i$, a compact set. Then there exists an open neighbourhood $f \in W$ such that for every $g \in W$ and $a \in X$ we have $|f(a) - g(a)| < \delta$, and we claim that this W is

as desired. Indeed, it is clear from the construction that every $g \in W$ has property U_ε, so all we have left to show is the statement regarding the diameter. Let $h = f{\restriction}_X + \delta \in K_C(X)$, and let $g \in W$. Then $\tilde{h}$ has property U_ε, by construction, and $g \leq \tilde{h}$. By Lemma 4.12 (and since our construction forces that $4\delta < \varepsilon$) we get $\tilde{h} - (\varepsilon - \delta) \leq g$. Thus $\operatorname{diam}(W) \leq \varepsilon - \delta < \varepsilon$, concluding the proof. $\blacksquare_{5.1}$

We now seek to show that types with property U are dense. The strategy is, first, characterise those $f \in K_C(E)$ for which f^* is continuous, and second, given such f modify it slightly to give it property U. For the first step, recall the following definition.

Definition 5.2. We say that a function $f \in K_C(E)$ is *local* if there are $f_k \in K_C(X_k)$, where each $X_k \subseteq E$ is convex and compact, such that $\tilde{f}_k \to f$ uniformly. The set of local functions in $K_C(E)$ was denoted in [Benb] by $K_{C,0}(E)$.

Lemma 5.3. *Let E be a finite-dimensional normed space, and let $f \in K_C(E)$. Then f is local if and only if f^* is continuous on $E^*_{\leq 1}$. Also, if f is isolated (so in particular, if it has property U) then it is local.*

Proof. Let first $X \subseteq E$ be compact, and in particular bounded, say $X \subseteq E_{\leq R}$, and let $g \in K_C(X)$. Then g^* is easily verified to be R-Lipschitz on its domain, and in particular continuous. Since a uniform limit of continuous functions is continuous, if f is local then f^* is continuous on $E^*_{\leq 1}$.

Conversely, assume that f^* is continuous on $E^*_{\leq 1}$. For each k let $f_k = f{\restriction}_{E_{\leq k}}$. Then $\tilde{f}_k \searrow f$ point-wise, and for $\lambda \in E^* \leq 1$ we have

$$f^*(\lambda) = \sup_v \lambda v - f(v) = \sup_{v,k} \lambda v - f_k(v) = \sup_k f_k^*(\lambda),$$

i.e., $f_k^* \nearrow f^*$ point-wise on $E_{\leq 1}$. Since each f_k^* is lower semi-continuous, f^* is continuous, and $E_{\leq 1}$ is compact, this implies that $f_k^* \to f^*$ uniformly on $E^*_{\leq 1}$, whereby $\tilde{f}_k \to f$ uniformly, and f is local.

Finally, assume that f is isolated, and let f_k be as above. Then for each $\varepsilon > 0$ there exists a neighbourhood $f \in W$ of diameter $\leq \varepsilon$. On the other hand, for every neighbourhood W of f there exists k such that $\tilde{f}_\ell \in W$ for all $\ell \geq k$. It follows that $\tilde{f}_k \to f$ uniformly, and f is local. $\blacksquare_{5.3}$

The converse of the last assertion fails. Indeed, take any local $f \in K_C(E)$ (isolated or not). Then for any $r > 0$, $f + r$ is still local, but cannot have property U, and by what we shall show later on, it follows that it is not isolated.

In fact, we shall only require the easy direction of Lemma 5.3, but given the crucial role played by local functions in [Benb] it seemed appropriate to give the full characterisation. The next sequence of technical lemmas will allow us to construct types with property U.

Lemma 5.4. *Let $f \colon E_{\leq 1} \to \mathbf{R}$ be convex, continuous, satisfying $f(v)+f(-v) < 0$ for all $v \in E_{=1}$. Then $f(v) + f(-v) < 0$ for all $v \in E_{\leq 1}$, and for any $v \in E_{=1}$ there exists $g \colon E_{\leq 1} \to \mathbf{R}$ convex, continuous, satisfying*

$$g(v) = \frac{2}{3}f(v) - \frac{1}{3}f(-v), \qquad f(u) \leq g(u) < \frac{1}{2}f(u) - \frac{1}{2}f(-u) \qquad \text{for } u \in E_{\leq 1}.$$

Moreover, given any $r < 1$, we may have g agree with f on $E_{\leq r}$.

Proof. The first assertion is immediate, so we prove the second. Let $\beta = f(v) + f(-v) < 0$. Since f is convex and lower semi-continuous, there exist functionals $\mu^\pm \in E^*$ such that $f(\pm u) > \mu^\pm(u - v) + f(\pm v) + \frac{\beta}{7}$ for all u. Let also λ norm v, and for $\alpha > 0$ define

$$h_\alpha(u) = \alpha(\lambda u - 1) + \frac{1}{2}\mu^+(u - v) + \frac{2}{3}f(v) - \frac{1}{2}\mu^-(u - v) - \frac{1}{3}f(-v).$$

Then, first, $h_\alpha(v) = \frac{2}{3}f(v) - \frac{1}{3}f(-v) > f(v)$. Second, for $u \in E_{\leq 1}$ we have $h_\alpha(u) \leq \frac{1}{2}f(u) - \frac{1}{2}f(-u) + \frac{\beta}{6} - \frac{\beta}{7} < \frac{1}{2}f(u) - \frac{1}{2}f(-u)$. Third, taking α big enough, we also have $h_\alpha \leq f$ on $E_{\leq r}$. Then $g = h_\alpha \vee f$ is as desired. $\blacksquare_{5.4}$

Lemma 5.5. *Let E be a finite-dimensional normed space, and let $f \colon E_{\leq 1} \to \mathbf{R}$ be convex, continuous, satisfying $f(v) + f(-v) < 0$ for all $v \in E_{=1}$. Then there exists $g \colon E_{\leq 1} \to \mathbf{R}$ convex, lower semi-continuous, satisfying $g(v) + g(-v) = 0$ for $v \in E_{=1}$, and in addition $g \geq f$.*

Moreover, given any $r < 1$, we may have g agree with f on $E_{\leq r}$.

Proof. First, for each $v \in E_{=1}$ apply Lemma 5.4, obtaining g_v. By continuity, there exists an open neighbourhood $v \in V_v$ such that $g_v(u) > \frac{3}{4}f(u) - \frac{1}{4}f(u)$ for all $u \in V_v \cap E_{=1}$. By compactness of the unit sphere in finite dimension we have $E_{=1} \subseteq \bigcup_{i<n} V_{v_i}$ for some finite family, and then let us define $f' = \bigvee_i g_{v_i}$. Then f' is convex, continuous, and for $v \in E_{=1}$ it satisfies

$$\frac{3}{4}f(v) - \frac{1}{4}f(-v) < f'(v) < \frac{1}{2}f(v) - \frac{1}{2}f(-v),$$

and therefore

$$\frac{1}{2}\big[f(v) + f(-v)\big] < f'(v) + f'(-v) < 0.$$

We may therefore construct an increasing sequence in this fashion: $f_0 = f$, $f_{k+1} = f'_k$, and define $g = \bigvee f_k$.

Then g is convex and lower semi-continuous, as supremum of such, and by construction $g(v)+g(-v) = 0$ on $E_{=1}$. Finally, we can do the entire construction so that each f_k, and therefore g, agree with f on $E_{\leq r}$. $\blacksquare_{5.5}$

Notice that by the last assertion of Lemma 5.3, and upon passing to the dual, such g is in fact continuous on $E_{\leq 1}$.

Lemma 5.6. *Let E be a finite-dimensional normed space. Then the set of $f \in K_C(E)$ with property U is dense.*

Proof. Let $V \subseteq K_C(E)$ be open and let $f \in V$. We may assume that V is basic, namely of the form $\big\{g \in K_C(E)\colon g(v_i) \in I_i \text{ for all } i < n\big\}$ for some $v_i \in E$ and open intervals I_i. Let $R = \max \|v_i\| + 1$, $f_0 = f{\restriction}_{E_{\leq R}}$ and $M = 2\sup f_0 + 1$, and for $\varepsilon > 0$ let $f_\varepsilon = (1-2\varepsilon)f_0 + \varepsilon M$. Then $f_\varepsilon \in K_C(E_{\leq R})$, and for ε small enough, which we now fix, we also have $\tilde{f}_\varepsilon \in V$. We observe that f_ε is $(1-2\varepsilon)$-Lipschitz, f_ε^* is continuous on $E_{\leq 1}$, and $f_\varepsilon^*(\lambda) + f_\varepsilon^*(-\lambda) < -\varepsilon$ for all $\lambda \in E^*_{=1}$.

By Lemma 5.5 (and Lemma 4.11) there exists $g \in K_C(E)$ with property U, such that $g^* \geq f_\varepsilon^*$ and the two agree on $E^*_{\leq 1-\varepsilon}$. By Corollary 4.9 for every $v \in E_{<R}$ there exists $\lambda \in E^*_{<1-\varepsilon}$ such that $f(v) + f^*(\lambda) = \lambda v$. By Lemma 4.10, applied to $X = E^*_{<1-\varepsilon}$, we have $g{\restriction}_{E_{<R}} = f_\varepsilon{\restriction}_{E_{<R}}$, so $g \in V$ as well, and we are done. $\blacksquare_{5.6}$

Theorem 5.7. *Let E be a finite-dimensional normed space. Then for every I, the isolated I-types over E are exactly the ones with property U, and they are topologically dense among all types. In particular, $\mathbf{G}[E]$ exists, and atomic extensions are exactly those with property U.*

Proof. First, let us consider 1-types. By Lemma 5.1, every 1-type with property U is isolated. On the other hand, by Lemma 5.6, the set of 1-types with property U is topologically dense in $S_1(E)$, and by Lemma 4.2 it is metrically closed. Therefore, by Lemma 1.11, every isolated 1-type has property U. By Lemma 2.14 and Lemma 4.3, the same is true for types in arbitrarily many variables. We conclude using Theorem 2.15. $\blacksquare_{5.7}$

6. Isolated types over infinite-dimensional spaces

Let us observe, first, that the hypothesis that $\dim E < \infty$ is not superfluous in Theorem 5.7. Indeed, let $E \subseteq F$ be Hilbert spaces with $\dim E = \infty$, and let $v \in F \setminus E$. Then, on the one hand, the extension has property U, while on the other hand, it is not difficult to check that $\mathrm{tp}(v/E)$ is not isolated. If $\mathrm{tp}(v/E) \in U \subseteq \mathrm{S}_1(E)$, we may replace U with a basic open set which only uses parameters in some finite-dimensional $E_0 \subseteq E$, and then U also contains $\mathrm{tp}(u/E)$ for some $u \in E_1$ where $E_1 \subseteq E$ is *any* proper extension of E_0. This suggests that over infinite-dimensional spaces the situation is quite different, and that there may be "few" isolated types. In fact, we shall show that there are none, except for the ones realised in E (which are always isolated). For this, we start by analysing the finite-dimensional case a little further. We shall use the following easy fact regarding the convex function "generated" by an arbitrary function.

Fact 6.1. *Let $X_0 \subseteq E$ be any set and let $X = \overline{\mathrm{Conv}(X_0)}$ be its closed convex hull. Let $f_0\colon X_0 \to \mathbf{R}$ be an arbitrary function such that $f \geq \lambda + \alpha$ for some linear functional λ and $\alpha \in \mathbf{R}$ (namely, such that f_0^* is not identically $+\infty$). Then f_0^{**} (the double convex conjugate) is the greatest lower semi-continuous convex function lying below f_0, and it can be recovered explicitly as*

$$f(x) = \inf\left\{\liminf \sum_i \alpha_{n,i} f_0(x_i)\colon \sum_i \alpha_{n,i} x_i \to x\right\},$$

the infimum being taken over all possible expressions of x as a limit of finite convex combinations of members of X_0 (so $f(x) = \inf \varnothing = \infty$ if $x \notin X$).

If f_0 is such that for every $x \in X_0$, and for every sequence $\sum_i \alpha_{n,i} x_i$ of finite convex combinations in X_0 which converges to x, we have $f_0(x) \leq \liminf_n \sum_i \alpha_i f_0(x_i)$, then f extends f_0. If f_0 is lower semi-continuous then this last condition holds automatically for every $x \in X_0$ which is an extreme point of X.

Lemma 6.2. *Let E be finite-dimensional and let $f \in K_C(E)$ have property U. Then for every $F \supseteq E$ there exists $g \in K_C(F)$ extending f which has property U as well, and such that $\inf g = \inf f$.*

Proof. Before we start, let us remark that since $E \subseteq F$, f can also be viewed as a convex function on F (extended by ∞ outside E), so the symbol f^* can represent two functions, one on E^* and one on F^*. In order to avoid confusion, we shall only use f^* to denote the former. Then the latter is merely the composition $f^*\pi$, where $\pi\colon F^* \to E^*$ is the restriction map.

Let $X \subseteq F^*_{\leq 1}$ denote the set of extreme points, so $F^*_{\leq 1} = \overline{\operatorname{Conv}(X)}$, and let $Y \subseteq F^*_{\leq 1}$ denote the set of λ such that $\|\lambda\| = \|\pi\lambda\|$. Define $h_0\colon X \cup Y \to \mathbf{R}$ by

$$h_0(\lambda) = \begin{cases} \frac{f^*(\pi\lambda) - f^*(-\pi\lambda)}{2} & \lambda \in X, \\ f^*(\pi\lambda) & \lambda \in Y, \end{cases}$$

observing that since f has property U, the two cases agree on $X \cap Y$, and both f^* and h_0 are continuous. Let also $g = h_0^*$ and $h = g^* = h_0^{**}$. Then h is lower semi-continuous, and $\operatorname{dom} h = F^*_{\leq 1}$. Assume now that $\lambda \in Y$ is the limit as $n \to \infty$ of finite convex combinations $\sum \alpha_{n,i}\lambda_i$, where $\lambda_i \in X \cup Y$. Then $\sum \alpha_{n,i} h_0(\lambda_i) \geq \sum \alpha_{n,i} f^*(\pi\lambda_i) \geq f^*\left(\sum \alpha_{n,i}\lambda_i\right) \to f^*(\lambda) = h_0(\lambda)$. By Fact 6.1, it follows that $h{\restriction}_{X\cup Y} = h_0$. By Lemma 4.11, $g \in K_C(F)$ and has property U.

Now let $v \in E$. On the one hand, $h_0 \geq f^*\pi$ implies $g(v) \leq f(v)$. On the other hand, for every $\lambda \in E^*_{\leq 1}$ there exists $\mu \in Y$ extending λ. Then $g(v) \geq \mu v - h_0(\mu) = \lambda v - f^*(\lambda)$ for all such λ, whereby $g(v) \geq f(v)$, so $g{\restriction}_E = f$.

Finally, since $0 \in Y$ we have $\inf g = -g^*(0) = -f^*(\pi 0) = \inf f$. $\blacksquare_{6.2}$

It follows that over infinite-dimensional E is as far as possible from the situation described in Theorem 5.7.

Theorem 6.3. *Let E be a Banach space (of arbitrary dimension).*

(i) *The set of types in $\mathrm{S}_n(E)$ with property U is dense. Moreover, the set of non realised types with property U is dense.*
(ii) *If $\dim E = \infty$ then the only isolated types over E are the realised ones.*

Proof. By Theorem 5.7, we may assume for both items that $\dim E = \infty$. Let us first prove this for 1-types. In fact, we shall prove that if $W \subseteq K_C(E)$ is open and $f \in W$ is not realised, namely $\inf f > 0$, then W contains a non realised type with property U and $\operatorname{diam}(W) \geq \inf f$.

We may assume that W is basic, and therefore defined using finitely many parameters in E. Let $E_0 \subseteq E$ be the subspace generated by these parameters, so W is the pull-back of some open $V \subseteq K_C(E)$, and $f{\restriction}_E \in V$. If $r < \inf f$, then possibly shrinking V, for every $g \in V$ we have $r < \inf g$. By Theorem 5.7 there exists $g \in V$ with property U. By Lemma 6.2, there exists $g_1 \in W$ extending g with property U, such that $\inf g_1 = \inf g > r$, so g_1 is not realised, concluding the proof of the first part. Let also $g_2 = \tilde{g} \in W$ be the trivial extension of g.

Since $\dim E = \infty$, there exist a linear functional $\lambda \in E^*$ such that $E_0 \subseteq \ker \lambda$ and $\|\lambda\| = 1$. Then on the one hand we have $g_1^*(\lambda) + g_1^*(-\lambda) = 0$, so we may assume that $g_1^*(\lambda) \geq 0$. On the other hand, if $\pi\colon E^* \to E_0^*$ is the projection map,

$$g_2^*(\lambda) = g^*(\pi\lambda) = g^*(0) = -\inf g < -r.$$

Thus $\operatorname{diam}(W) \geq \|g_1 - g_2\| = \|g_1^* - g_2^*\| > r$, and this for every $r < \inf f$, proving the second part.

For n-types, the second item follows immediately from the case $n = 1$. As for the first item, we proceed by induction, the case $n = 1$ having been proved. Let $W \subseteq \mathrm{S}_{n+1}(E)$ be open and non empty, and let $V \subseteq \mathrm{S}_n(E)$ be its projection on the first n variables. By Lemma 1.14, V is open as well, so by the induction hypothesis it contains a non realised type ξ with property U. Let W_ξ be the fibre of W over ξ. Then it is open and non empty as well, and can be identified with an open set $W' \subseteq \mathrm{S}_1(E[\xi])$. By the case $n = 1$, there exists $\zeta \in W'$ with property U. Let $\bar{v}$ and u be the respective realisations of ξ and ζ in $E[\xi][\zeta]$. Then $\rho = \operatorname{tp}(\bar{v}, u) \in W_\xi$ is the point corresponding to $\zeta \in W'$, and it is clear that $\rho \in W$ is non realised and has property U. $\blacksquare_{6.3}$

Corollary 6.4. *Let E be a separable Banach space. Then $\mathbf{G}[E]$ exists if and only if $\dim E < \infty$ or $E = \mathbf{G}$.*

Proof. For one direction, if E is a Gurarij space then E *is* $\mathbf{G}[E]$, and if $\dim E < \infty$ apply Theorem 5.7. Conversely, assume that $\mathbf{G}[E]$ exists and $\dim E = \infty$. Then the isolated types over E are dense, but by Theorem 6.3 these types are all realised. By Lemma 2.2, E is a Gurarij space. $\blacksquare_{6.4}$

This, together with Corollary 2.16, gives a full characterisation of generic orbits of the action $\mathrm{Aut}(\mathbf{G}) \curvearrowright \mathrm{Emb}(E, \mathbf{G})$.

Corollary 6.5. *Let E be a separable Banach space, and let the Polish group $H = \mathrm{Aut}(\mathbf{G})$ act continuously by composition on the Polish space of linear isometric embeddings $X = \mathrm{Emb}(E, \mathbf{G})$.*

(i) *All orbits are dense.*
(ii) *If $\dim E \leq \infty$ or $E \cong \mathbf{G}$ then $E \curvearrowright X$ admits a unique dense G_δ orbit. If $E = \mathbf{G}$, this orbit is $H \subseteq X$.*
(iii) *Otherwise, all orbits are meagre.*

Corollary 6.6. *Let E be a Banach space, and let κ be an infinite cardinal, greater or equal to the density character of E. Then there exists a Gurarij space G such that $E \subsetneq G$ is a proper extension with property U, and the density character of G is κ.*

Proof. By transfinite induction on $\alpha \leq \kappa$ we construct an increasing sequence E_α, where $E_0 = E$ and for limit ordinals, E_α is the completion of $\bigcup_{\beta<\alpha} E_\beta$. At successor steps we choose some non empty open set $U_\alpha \subseteq \mathrm{S}_1(E_\alpha)$, and let $E_{\alpha+1} = E_\alpha[\xi]$ where $\xi \in U_\alpha$ is not realised in E_α and has property U. We can do this in such a fashion that every non empty open set $V \subseteq \mathrm{S}_1(E_\kappa)$ contains the pull-back of some U_α, so by Lemma 2.2 $G = E_\kappa$ is a Gurarij space. The extension $E \subseteq G$ has property U by an easy induction argument, and it clearly has the desired density character. $\blacksquare_{6.6}$

Question 6.7. Is the set of types with property U over a separable E a G_δ set? It is when $\dim E < \infty$ or when $E \cong \mathbf{G}$.

Notice that for separable E for which this is true, we can give an alternative proof for Corollary 6.6 using the Omitting Types Theorem (and the fact that the types with property U are metrically closed).

Question 6.8. Let E be separable and consider all the separable Gurarij space extensions of E which have property U. If $\dim E < \infty$, we know that all these extensions are isometrically isomorphic over E, and that every Gurarij extension of E contains, as a sub-extension, one with property U. What of this, if any, survives when $\dim E = \infty$?

We saw that such extension at least exist, and since $\mathbf{G}$ admits a proper Gurarij space extension with property U, such extensions are not, in general, unique. Can one show that they are never unique? (If every Gurarij space extension of E contains a sub-extension with property U, then non uniqueness follows via a type omission argument). Are the Gurarij extensions with property U almost unique for some reasonable notion of "almost uniqueness" (e.g., almost isometrically unique)?

7. Counting types

We conclude with a calculation of the size of the type-space over a separable Banach space E. By "size" we mean here its metric density character (the cardinal $|\mathrm{S}_n(E)|$ is the continuum as soon as $n > 0$ and $E \neq 0$).

Theorem 7.1. *Let E be a separable Banach space.*

(i) *If E is finite-dimensional and polyhedral then $\mathrm{S}_n(E)$ is metrically separable.*
(ii) *Otherwise, $\mathrm{S}_n(E)$ has metric density character equal to the continuum for every $n \geq 1$.*

Proof. Assume first that E is finite-dimensional and polyhedral. Then by Melleray [Mel07, Remarks following Corollary 4.6], the space $K(E)$ is separable, and *a fortiori* so is $\mathrm{S}_1(E) = K_C(E)$. The passage from 1-types to n-types is done as in the proof of Lemma 2.14, and is left to the reader.

Now assume that E is not so. Then by Lindenstrauss [Lin64, Theorem 7.7] there exists a sequence $\{v_n\} \subseteq E$ such that for any $n \neq m$ and choice of signs:

$$\|v_n \pm v_m\| \leq \|v_n\| + \|v_m\| - 1.$$

Embed E (isometrically) in ℓ_∞, and for a sequence $\bar{\varepsilon} \in \{\pm 1\}^{\mathbf{N}}$, consider the family of closed balls $\overline{B}(\varepsilon_n v_n, \|v_n\| - \frac{1}{2})$. By hypothesis every two such balls intersect at a non empty set, and therefore there

exists $v \in \ell_\infty$ which belongs to them all. In other words, there exists $\xi_{\bar{\varepsilon}} = \mathrm{tp}(v/E) \in \mathrm{S}_1(E)$ such that $\|x - \varepsilon_n v_n\|^{\xi_{\bar{\varepsilon}}} \leq \|v_n\| - \frac{1}{2}$. If $\bar{\varepsilon} \neq \bar{\varepsilon}'$ then $d(\xi_{\bar{x}}, \bar{\varepsilon}') \geq 1$, so the density character of $\mathrm{S}_1(E)$ is at least the continuum. The same holds *a fortiori* for $\mathrm{S}_n(E)$, $n \geq 1$. $\blacksquare_{7.1}$

Remark 7.2. Lindenstrauss's argument is quite elementary and yields a quick proof for Theorem 7.1(ii) which does not depend on the machinery developed in earlier sections. Arguments closer to the spirit of the present paper can also be given.

First, let X_0 be the set of extreme points in $E^*_{\leq 1}$, and let Ξ be the set of lower semi-continuous functions $f_0 \colon X_0 \to \mathbf{R}$ which satisfy in addition $f_0(\lambda) + f_0(-\lambda) \leq 0$. Then E is not a finite-dimensional polyhedral space if and only if X_0 is infinite, in which case Ξ has density character continuum. If $f_0 \in \Xi$ and $f = f_0^{**}$ as in Fact 6.1 then $f{\restriction}_{X_0} = f_0$ and $f(\lambda) + f(-\lambda) \leq 0$ throughout $E^*_{\leq 1}$, so $f^* \in K_C(E)$ and we are done. Notice that this argument has the advantage of treating the two cases of "finite-dimensional, non polyhedral" and "infinite-dimensional" in the same manner, while the proof of [Lin64, Theorem 7.7] treats them separately, with the second one being significantly more involved.

Second, in the case where E is infinite-dimensional, Theorem 7.1(ii) is a special case of a general principle which may be worth a mention as well. This principle, akin to the fact that the cardinal of a perfect set is at least the continuum, says that if isolated types are not dense in $\mathrm{S}_n(E)$ (as is the case per Theorem 6.3) then the metric density character of $\mathrm{S}_n(E)$ must be at least the continuum. The general argument is as follows. First, for $r > 0$, let $X_r \subseteq \mathrm{S}_n(E)$ be the union of all open sets of diameter $\leq r$. If X_r is dense for every r then in every open set U one can find a sequence $\xi_n \in X_{2^{-n}}$ which converges metrically to some $\xi \in U$, and by Lemma 1.16 ξ is isolated, so the isolated types are dense after all. Therefore, for some $r > 0$, which we now fix, X_r is not dense (in our case, X_r is not dense for any $r > 0$; notice also that $\mathrm{S}_n(E)$ need not be metrisable, so we cannot use the Baire Category Theorem). We also fix an open set U disjoint from X_r. Since every open subset of U has diameter $> r$, and since the metric on $\mathrm{S}_n(E)$ is lower semi-continuous, one can build a binary tree of open subsets U_s, $s \in 2^{<\omega}$, such that $U_\varnothing = U$, $\overline{U_{s\frown i}} \subseteq U_s$ for $i = 0, 1$, and $d(U_{s\frown 0}, U_{s\frown 1}) > r$. Then for $\sigma \in 2^{\mathbf{N}}$ we have $\bigcap_n U_{\sigma{\restriction}_n} \neq \varnothing$, and these intersections all have distance $> r$ from one another, as desired.

This answers a Problem 2 of Avilés et al. [ACC+11, Section 4] in the negative (and we thank Wiesław Kubiś for having pointed this out to us). They say that a Banach space G is *of universal disposition for finite-dimensional spaces* if it satisfies a strengthening of Definition 2.1 with ψ being an isometry.

Corollary 7.3. *The density character of any space of universal disposition for finite-dimensional spaces is at least the continuum. In other words, the answer to Problem 2 of* [ACC+11, Section 4] *is negative.*

Proof. Assume that G is of universal disposition for finite-dimensional spaces. Then the Euclidean plane E embeds isometrically in G, and all types over E are realised in G, so the density character of G must be at least the metric density character of $S_1(E)$, namely the continuum. $\blacksquare_{7.3}$

On the other hand, say that a Gurarij space G is *strongly $\aleph_1$-homogeneous* if the following stronger version of Corollary 2.8 holds in G:

> For every separable $F \subseteq G$ and isometric embedding $\varphi \colon F \to G$ there exists an isometric automorphism $\psi \in \mathrm{Aut}(G)$ extending φ.

Clearly, a strongly $\aleph_1$-homogeneous Gurarij space is of universal disposition for finite-dimensional (and even separable) spaces. Moreover, there does exist such a space of density character the continuum. This is merely a special case of a general model theoretic result: for any cardinal κ and structure $\mathbf{M}$ of density character $\leq 2^\kappa$, in a language of cardinal $\leq \kappa$, there exists an elementary extension $\mathbf{M}' \succeq \mathbf{M}$ of density character still $\leq 2^\kappa$, which is moreover κ^+-saturated and strongly κ^+-homogeneous. Apply this to $\mathbf{M} = \mathbf{G}$ and $\kappa = \aleph_0$.

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